UNIVERSITY OF GLASGOW

DEPARTMENT OF AERONAUTICS AND FLUID MECHANICS

NON-LINEAR DIFFERENTIAL EQUATIONS HAVING CUBIC STIFFNESS TERMS

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A.W. Babister, M.A., Ph.D.

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SUMMARY

The nature of solutions of the autonomous equation

$$\ddot{x} + (b_0+b_2x^2)\dot{x} + c_1x + c_3x^3 = 0$$

is considered. Trajectories in the $(x,\dot{x})$ phase plane are given for various combinations of signs of the b and c parameters. It is well known that limit cycles can occur if c_1 and c_3 are both positive. Limit cycles are shown to occur if c_1 and c_3 have opposite signs, and their stability is investigated.

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General Introduction

This report is one of a series dealing with dynamical systems satisfying second-order non-linear differential equations (see Babister, 1972 and 1973). In this report we consider the nature of solutions of differential equations of the form

$$\frac{d^2x}{dt^2} + f(x)\frac{dx}{dt} + g(x) = 0 \qquad (1)$$

where

$$f(x) = b_0 + b_2x^2 \qquad (2)$$

and

$$g(x) = c_1x + c_3x^3 \qquad (3)$$

where b_0, b_2, c_1 and c_3 are real constants.

For given values of these parameters, the solution of (1) depends only on the initial conditions. Here x can be thought of as the displacement at time t. We shall consider both the variation of x with t and also the trajectories in the $(x,\dot{x})$ phase plane. Equation (1) thus relates to the free vibration of a dynamical system with a cubic restoring force and a non-linear damping force.

We shall be particularly interested in those cases in which self-sustained oscillations or limit cycles occur. If $c_3=0$ and b_0 and b_2 have opposite signs, equation (1) reduces to the Van der Pol equation, which is a familiar example of an autonomous system that possesses limit cycles. We shall show that limit cycles are also possible if $c_3 \neq 0$. Physical examples of such oscillations (or closely related ones) occur in the analysis of the transient response of large amplitude motions of aircraft (Shinbrot, 1954) and missiles (Murphy, 1960).

There are a number of very general theorems concerning the properties of autonomous systems such as that defined by (1) (see Sansone and Conti, 1964, and Levinson and Smith, 1942). However it is generally assumed that $g(x)$ is an odd function of x with $xg(x) > 0$ for all $x \neq 0$. Such a condition does not necessarily apply in the pitching motion of an aircraft. We shall therefore consider the nature of solutions of

$$\frac{d^2x}{dt^2} + (b_0+b_2x^2)\frac{dx}{dt} + c_1x + c_3x^3 = 0 , \qquad (4)$$

or the equivalent plane autonomous system

$$\left.\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= -(b_0+b_2x^2)y - c_1x - c_3x^3 \end{aligned}\right\} \qquad (5)$$

for all combinations of sign of the b and c parameters.

Any periodic solution of (5) is represented by a closed curve in the phase plane, and this curve intersects the x axis twice (at points corresponding to the maximum and minimum values of x on the closed curve). Also, since x and y are related by (5), if a trajectory in the phase plane does not cut the x axis, it must be an open trajectory, which starts at infinity at $t = -\infty$ and ends at a point at infinity at $t = +\infty$.

For the system considered, the point $x=0$ is an equilibrium position. In the $(x,\dot{x})$ phase plane, the origin is a singular point. By considering the signs of the coefficients in (4), we find that this point is a focus (or node) if $c_1 > 0$ and a saddle point if $c_1 < 0$. The focus at 0 will be stable or unstable according as $b_0 >$ or < 0. As shown by Loud (1964), the focus becomes a centre if $b_0 = b_2 = 0$ (with $c_1 > 0$).

If both c_1 and c_3 have the same sign, this is the only real singular point. However, if $c_1 c_3 < 0$, (5) has two other singular points at $x = \pm\sqrt{-c_1/c_3}$. On putting $x = z + \gamma$, (4) becomes.

$$\frac{d^2z}{dt^2} + \left[b_0+b_2\gamma^2+2b_2\gamma z+b_2z^2\right]\frac{dz}{dt} + c_1\gamma + c_3\gamma^3 + (c_1+3c_3\gamma^2)z + 3c_3\gamma z^2 + c_3z^3 = 0. \qquad (6)$$

If $\gamma^2 = -c_1/c_3$, (6) reduces to

$$\frac{d^2z}{dt^2} + \left[b_0-(b_2c_1/c_3)+2b_2\gamma z+b_2z^2\right]\frac{dz}{dt} - 2c_1z + 3c_3\gamma z^2 + c_3z^3 = 0. \quad (7)$$

From (7) we see that the points $x=\pm\sqrt{-c_1/c_3}$ will both be foci (or nodes) if $c_1 < 0$ and will both be saddle points if $c_1 > 0$. These foci will be stable or unstable according as $b_0-(b_2c_1/c_3) >$ or < 0. More generally, from (6), we see that any equation of the form

$$\frac{d^2z}{dt^2} + (B_0+B_1z+B_2z^2)\frac{dz}{dt} + C_0 + C_1z + C_2z^2 + C_3z^3 = 0, \quad (8)$$

in which B_2 and C_3 are not both zero, can be put in the same form as (4) with real coefficients on letting $x = z+\gamma$ where

$$\gamma = B_1/2B_2 = C_2/3C_3$$

and

$$C_0 - C_1\gamma + C_2\gamma^2 - C_3\gamma^3 = 0.$$

In (4), put $x = \alpha X$, $t = \beta T$, where α and β are constants.

Then

$$\frac{d^2X}{dT^2} + (\beta b_0+\alpha^2\beta b_2X^2)\frac{dX}{dT} + \beta^2c_1X + \alpha^2\beta^2c_3X^3 = 0. \quad (9)$$

Thus, if (4) has the solution $x = \phi(t)$, with $x=x_0$, $y=y_0$ at $t=0$, (9) has the solution $X = \alpha^{-1}\phi(\beta T)$, with $X=x_0/\alpha$ and $dX/dT = \beta y_0/\alpha$ at $T = 0$. Thus, if (4) has a periodic solution, (9) with any non-zero α and β will also have a periodic solution. We note that a variation in the value of α merely affects the non-linear terms in (9). If α is replaced by $-\alpha$, (9) is unchanged but the initial values of X and dX/dT both change sign. Thus any solution of (9) has a corresponding solution in which, at any time T, the value of X is of equal magnitude but of opposite sign; thus, in the phase plane, if $x(t)$, $y(t)$ is a solution of (5), so is $-x(t)$, $-y(t)$. In particular, if (5) has a periodic solution, which is represented by a closed curve encircling the origin and passing through the point (x,y) in the phase plane, it is readily shown by topological argument that the point $(-x, -y)$ also lies on this curve (Lefschetz, 1957). Thus any closed curve for the system (5), which encompasses the origin, will have equal and opposite intercepts on the x axis (and also on the y axis); as we shall show, for certain values of the parameters b and c, (5) can have periodic solutions which do not encompass the origin. Again, if $\alpha=1$ and $\beta= -1$, the coefficients b_0 and b_2 in (4) become $-b_0$ and $-b_2$, and the variation of X with T is identical with that of x with $(-t)$. Thus the positive semi-trajectory $(T > 0)$ in the X plane is the same as the negative semi-trajectory $(t<0)$ in the x plane.

Scaling factors were used in the numerical solutions given in this report, many of which were calculated on Glasgow University's analogue computer (PACE). The computer calculations were carried out for the equation

$$0.01 \frac{d^2X}{dT^2} + (0.01\ b_0+0.16b_2X^2)\frac{dX}{dT} + 0.01\ c_1X + 0.16\ c_3X^3 = 0 \qquad (10)$$

with the numerical values of the b and c parameters being less than or equal to 1. Thus the solutions were performed in real time ($\beta=1$) with a scaling factor $\alpha=4$. The magnitudes of the initial conditions were then never greater than unity.

In part 1 of this report we discuss the nature of solutions of the differential equation (4) with $c_3=0$, and in part 2 we deal with the case $c_3\neq0$. For ease of reference the various cases are set out in table 1.

TABLE 1

Index to discussion of solutions of

$$\ddot{x} + (b_0 + b_2x^2)\dot{x} + c_1x + c_3x^3 = 0$$

Para.	Case	b_2	c_1	c_3	General remarks
			PART ONE		
1.2	1	0	0	0	
1.2	2	+	0	0	
1.2	3	-	0	0	
1.3	4	0	+	0	Periodic solutions
1.3	5	+	+	0	Some limit cycles
1.3	6	-	+	0	
1.4	7	0	-	0	
1.4	8	+	-	0	
1.4	9	-	-	0	
			PART TWO		
2.2	10	0	0	+	Periodic solutions
2.2	11	+	0	+	Some limit cycles
2.2	12	-	0	+	
2.3	13	0	0	-	
2.3	14	+	0	-	
2.3	15	-	0	-	
2.4	16	0	+	+	Periodic solutions
2.4	17	+	+	+	Some limit cycles
2.4	18	-	+	+	
2.5	19	0	+	-	Periodic solutions
2.5	20	+	+	-	Some limit cycles
2.5	21	-	+	-	
2.6	22	0	-	+	Periodic solutions
2.6	23	+	-	+	Some limit cycles
2.6	24	-	-	+	
2.7	25	0	-	-	
2.7	26	+	-	-	
2.7	27	-	-	-	

PART 1

Non-linear differential equations of the form $\ddot{x} + (b_0+b_2x^2)\dot{x} + c_1x = 0$

1.1 Introduction

We consider solutions of the equation

$$\ddot{x} + (b_0+b_2x^2)\,\dot{x} + c_1x = 0 \tag{11}$$

or the equivalent system

$$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\,(b_0+b_2x^2)\,y - c_1x \end{aligned}\right\} \tag{12}$$

where b_0, b_2 and c_1 are real constants. In particular we shall show how the nature of the solution depends upon the initial conditions $x=x_0$, $y=\dot{x}_0 = y_0$ at time $t=0$.

1.2 Systems with zero stiffness $(c_1=0)$

Case 1 $\quad b_2=0,\quad c_1=0$

$$\ddot{x} + b_0\dot{x} = 0. \tag{13}$$

Equation (13) is a linear differential equation. The general solution and the trajectories in the phase plane were discussed in the previous report (Babister, 1973).

Case 2 $\qquad b_2 > 0, \quad c_1 = 0$

$$\ddot{x} + (b_0 + b_2 x^2)\dot{x} = 0. \tag{14}$$

Equation (14) has a first integral

$$\dot{x} = -\frac{1}{3} b_2 F(x) = -\frac{1}{3} b_2 \left[x^3 + 3\frac{b_0}{b_2} x - A^3 \right] \tag{15}$$

where A is a constant.

If $A \neq 0$, we factorise the r.h.s. of (15) in the form

$$F(x) = x^3 + 3\frac{b_0}{b_2} x - A^3 = (x - A\theta)\left(x^2 + A\theta x + \frac{A^2}{\theta}\right) \tag{16}$$

where

$$A^2 \theta^3 + 3\frac{b_0}{b_2}\theta - A^2 = 0, \tag{17}$$

θ being real.

From (16), if $0 < \theta^3 < 4$, the cubic equation $F(x)=0$ has only one real root $x=A\theta$. The general solution of (15) is then

$$(b_0 + A^2\theta^2 b_2)(t+B) = -\log|x - A\theta| + \tfrac{1}{2}\log\left(x^2 + A\theta x + \frac{A^2}{\theta}\right)$$

$$+ \frac{3\theta\sqrt{\theta}}{\sqrt{4-\theta^3}} \tan^{-1} \frac{(2x + A\theta)\sqrt{\theta}}{A\sqrt{4-\theta^3}} \tag{18}$$

where B is a constant. As can be seen from (16), this solution always holds if $b_0/b_2 > 0$ and $A \neq 0$. If

$b_0 > 0$, $b_2 > 0$ and $A = 0$, the general solution of (15) is

$$b_0(t+B) = -\log|x| + \tfrac{1}{2}\log\left(x^2 + 3\frac{b_0}{b_2}\right). \tag{19}$$

If $b_0=0$ and $A\neq 0$, from (17), $\theta = 1$ and the general solution of (15) is

$$A^2 b_2(t+B) = -\log|x-A| + \tfrac{1}{2}\log(x^2+Ax+A^2) + \sqrt{3}\tan^{-1}\frac{2x+A}{\sqrt{3}A}. \tag{20}$$

If $b_0=0$ and $A=0$, the general solution of (15) is

$$b_2(t+B) = 3/(2x^2) \tag{21}$$

If $b_0/b_2 < 0$, the cubic equation $F(x)=0$ can have one or three real roots, depending upon the value of A (or θ). This is shown in figure 1, in which the variation of $F(x)$ with x is shown for $b_0/b_2 = -1$. If $(-b_2A^2/b_0) > 4^{1/3}$ the cubic still has only one real root and equation (18) is still valid.

If $(-b_2A^2/b_0) = 4^{1/3}$ from (17), $\theta^3 = 4$ or $-\frac{1}{2}$. With $\theta=4^{1/3}$, (16) becomes

$$F(x) = (x-A\theta)(x+\tfrac{1}{2}A\theta)^2$$

and the general solution of (15) is then

$$3b_0(t+B) = \log|x-A\theta| - \log|x+\tfrac{1}{2}A\theta| + \frac{3A\theta}{2x+A\theta} \tag{22}$$

where B is a constant and $\theta = 4^{1/3}$.

If $0 < (-b_2A^2/b_0) < 4^{1/3}$, the cubic equation $F(x) = 0$ has three real roots $A\theta$, $A\theta_1$ and $A\theta_2$, with $\theta > 4^{1/3}$ and θ_1 and θ_2 both negative. The general solution of (15) is then

$$(b_0+A^2\theta^2b_2)\ (t+B) = -\log|x-A\theta| + \frac{\theta-\theta_2}{\theta_1-\theta_2}\log|x-A\theta_1| + \frac{\theta-\theta_1}{\theta_2-\theta_1}\log|x-A\theta_2|\ . \tag{23}$$

We see from (16) that θ_1 and θ_2 are the roots of

$$\lambda^2 + \theta\lambda + \frac{1}{\theta} = 0. \tag{24}$$

If $b_0 < 0$, $b_2 > 0$ and $A=0$, the general solution of (15) is

$$b_0(t+B) = -\log|x| + \tfrac{1}{2}\log\left|x^2 + 3\frac{b_0}{b_2}\right|. \tag{25}$$

From (15) we see that the trajectories in the phase plane are arcs of the cubic curves

$$y + b_0x + \frac{1}{3}b_2x^3 = \frac{1}{3}b_2A^3\ , \tag{26}$$

as shown in figs. 2 - 4, in which y/b_2 is plotted against

x for $b_0/b_2 = 0$, 1 and -1. If $y_0 = 0$, x will remain constant for all values of t; the phase plane curve is then a point. Thus there is a whole line of equilibrium points along the x axis. In general from (26), when $x = 0$, $y = \frac{1}{3} b_2 A^3$. From (18) - (21), if $b_0 \geqslant 0$ and $y_0 \neq 0$, $x \rightarrow A\theta$ as $t \rightarrow \infty$. If $b_0 < 0$ and $y_0 \neq 0$, x will tend to one of the roots of $F(x) = 0$ as $t \rightarrow \infty$. As shown in figure 4, some trajectories may be of finite length, commencing at one root of $F(x) = 0$ and ending at another root (this root being either the largest or the smallest root of $F(x) = 0$). Along a given trajectory, y $(=\dot{x})$ does not change sign. We note that the solutions are bounded as $t \rightarrow \infty$ for all values of b_0/b_2.

Case 3 $\qquad b_2 < 0$, $c_1 = 0$

$$\ddot{x} + (b_0 + b_2 x^2)\dot{x} = 0. \tag{27}$$

On putting $t = -T$, (27) becomes

$$\frac{d^2x}{dT^2} - (b_0 + b_2 x^2)\frac{dx}{dT} = 0 \tag{28}$$

which is of the same form as (14). Thus the integral curves for this case are found by applying a time scaling factor $\beta = -1$ to those of case 2. The trajectories in the phase plane are given by (26), the point $(x, y/|b_2|)$ in figures 2-4 being transformed into the point $(x, -y/|b_2|)$

for systems having the same values of b_0/b_2; in addition, the arrows on the curves should be reversed. Thus, in this case, the solutions of (27) are unbounded and divergent as $t \to \infty$, except for those cases in which $F(x)=0$ has three real roots; certain trajectories (of relatively small amplitude) then tend to a finite value (corresponding to that root which is neither the largest nor the smallest of $F(x)=0$).

1.3 Systems with damping and positive stiffness

Case 4 $\qquad b_2 = 0, \; c_1 > 0$

$$\ddot{x} + b_0\dot{x} + c_1 x = 0. \tag{29}$$

Equation (29) is a linear differential equation. The general solution and the trajectories in the phase plane were discussed in the previous paper (Babister, 1973).

Case 5 $\qquad b_2 > 0, \; c_1 > 0$

$$\ddot{x} + (b_0+b_2x^2)\dot{x} + c_1 x = 0. \tag{30}$$

This is a particular example of Lienard's equation

$$\ddot{x} + f(x)\dot{x} + c_1 x = 0 \tag{31}$$

in which $f(x)$ is continuous for all finite x and c_1 is a positive constant. The existence of periodic solutions of (30) or (31) is equivalent to the existence of cycles in the phase plane.

In general (30), or (31), has no first integral, but some very general results are known relating to solutions of these equations, or to those of the equivalent system

$$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= - f(x)y - c_1 x \end{aligned}\right\} \quad (32)$$

For such a system all the points of the (x,y) phase plane are regular, with the exception of the origin.

If $f(x)$ is identically zero, the curves in the phase plane are concentric ellipses, given by

$$c_1 x^2 + y^2 = \text{constant}; \quad (33)$$

none of the curves is a limit cycle.

If $f(x) \neq 0$, the slope of the curves in the phase plane is given by

$$\frac{dy}{dx} = - f(x) - c_1 \frac{x}{y} \quad (34)$$

From (34) we see that, if $f(x)$ is everywhere positive (except, possibly, for $x=0$), all the curves in the phase plane cross the curves of the family of ellipses (33) in the inward direction, and approach the origin as a limit as $t \to \infty$. This is shown for equation (30) in figure 5, in which $y/\sqrt{c_1}$ is plotted against x for $b_0=0$, $b_2/\sqrt{c_1} = 1$. We see that the curves spiral round 0; every motion is damped out and no periodic motion exists. Similarly if $f(x)$ is everywhere negative (except, possibly, for $x=0$), all trajectories go to infinity as $t \to \infty$.

If $f(x)$ is not of invariable sign, self-sustained oscillations may occur. Thus, if $b_0 < 0$ and $b_2 > 0$ (with $c_1 > 0$), on putting $x = \alpha X$, $t = \beta T$, where $\alpha = \sqrt{- b_0/b_2}$ and $\beta = 1/\sqrt{c_1} > 0$, (30) reduces to Van der Pol's equation

$$\frac{d^2X}{dT^2} + \frac{b_0}{\sqrt{c_1}} (1 - X^2) \frac{dX}{dT} + X = 0, \tag{35}$$

which has one limit cycle, which is stable if $b_0 < 0$. This is a closed trajectory (or Poisson stable trajectory) surrounding the singular point at 0, as shown for equation (30) in figure 6, in which $\sqrt{-b_2/b_0 c_1}\, y$ is plotted against $\sqrt{-b_2/b_0}\, x$ for $b_0/\sqrt{c_1} = -1$. We note that, here, $f(x)$ is negative for $\sqrt{-b_2/b_0}\, |x| < 1$ and positive for $\sqrt{-b_2/b_0}\, |x| > 1$; that is, the system has negative damping for $\sqrt{-b_2/b_0}\, |x| < 1$. The limit cycle divides the phase plane into two regions. Trajectories which start near the origin spiral away from it towards the limit cycle; those which start far from the origin also spiral in towards the same cycle. Thus all motions of the system tend with increasing time to a single periodic solution, represented by the regular limiting trajectory.

If c_1 is positive $(=\omega^2)$ and the other parameters of (30) are small, the oscillatory behaviour of the system can be examined by Krylov and Bogoulioubov's method of the first approximation. We write the solution of (30) in the form

$$x = A \sin \chi \tag{36}$$

where $\chi = \omega t + \phi$, and A and ϕ are both functions of t. It can be shown that if the changes in the amplitude A and phase ϕ during a cycle are taken to be small, the rate of change of A with time is given by

$$\dot{A} = -\tfrac{1}{2} A \left(b_0 + \frac{1}{4} b_2 A^2\right). \tag{37}$$

From (37), when the steady state is reached $\dot{A} = 0$ and thus $A = 0$ or $2\sqrt{-b_0/b_2}$. Thus if b_0 and b_2 are of opposite sign (and small compared with c_1) the system (30) will have a limit cycle in the $(x, \dot{x})$ phase plane of amplitude $2\sqrt{-b_0/b_2}$. More generally, from (30), it can be shown that the equation of the limit cycle is of the form

$$\sqrt{-\frac{b_2}{b_0 c_1}}\, y = f\left(\sqrt{-\frac{b_2}{b_0}}\, x,\ \frac{b_0}{\sqrt{c_1}}\right). \tag{38}$$

From figure 6 we see that, if $b_0/\sqrt{c_1}$ is not small, the limit cycle departs from the near-circular shape, but the maximum and minimum x amplitudes are still approximately

$2\sqrt{-b_0/b_2}$. As shown by Urabe, the x amplitude of the limit cycle varies little even with extremely high damping, the cycle becoming more elongated in the y direction (for a given value of b_0/b_2); on the y axis, the slope of the phase plane trajectories is $dy/dx = -b_0$. The variation of the y amplitude with damping is shown in figure 7 (which is based on Urabe's results); it can be seen that at high values of the damping, $\sqrt{-(b_2/b_0\ c_1)}\ y_{max}$ increases almost linearly with $b_0/\sqrt{c_1}$. For such values of the damping, the maximum and minimum y amplitudes are approximately $\mp 1.4\ b_0\sqrt{-b_0/b_2}$.

The method of the first approximation shows that, for small values of the damping, the phase ϕ is constant and thus the period T is $2\pi/\sqrt{c_1}$. Urabe has shown that the period increases monotonically with the damping. For high values of the damping, the period T is approximately $1.6\ b_0/c_1$. The periodic solution is stable provided that $b_0 < 0$ and $b_2 > 0$.

<u>Case 6</u> $b_2 < 0$, $c_1 > 0$.

$$\ddot{x} + (b_0 + b_2 x^2)\dot{x} + c_1 x = 0. \qquad (39)$$

On putting $t = -T$, (39) becomes

$$\frac{d^2x}{dT^2} - (b_0+b_2x^2)\frac{dx}{dT} + c_1x = 0 \qquad (40)$$

which is of the same form as (30). The integral curves for this case are found by applying a time scaling factor $\beta = -1$ to those of case 5. If $b_0 \leq 0$, with $b_2 < 0$, all the phase plane trajectories diverge to infinity as $t \to \infty$. If $b_0 > 0$ and $b_2 < 0$ (with $c_1 > 0$), there is one limit cycle, which is unstable. The cycle is a closed trajectory and forms the boundary of a simply connected domain in the phase plane. Trajectories starting at points within this domain spiral in to the origin (the damping is always positive if $\sqrt{-b_2/b_0}\ |x| < 1$), whereas all trajectories in the region outside this domain diverge to infinity.

The self-sustained oscillations we have just considered in cases 5 and 6 also occur in systems in which the damping coefficient involves $\dot{x}$ and $\dot{x}^3$. Thus the system

$$\ddot{z} + b_0\dot{z} + \frac{1}{3}b_2\dot{z}^3 + c_1z = 0 \qquad (41)$$

becomes identical with (30) on differentiating with respect to t and putting $x = \dot{z}$. More generally, any equation of the form

$$\ddot{z} + F(\dot{z})\,\dot{z} + c_1z = 0 \qquad (42)$$

can be reduced to Lienard's equation (31). On putting $z = \alpha Z$, $t = \beta T$, where $\alpha^2 = -b_0/b_2c_1$ and $\beta^2 = 1/c_1$, (41) reduces to Rayleigh's equation.

$$\frac{d^2Z}{dT^2} + \frac{b_0}{\sqrt{c_1}} \left\{ \frac{dZ}{dT} - \frac{1}{3}\left(\frac{dZ}{dT}\right)^3 \right\} + Z = 0, \tag{43}$$

which has one limit cycle which is stable if $b_0 < 0$. As above, this is a closed trajectory surrounding the singular point at 0. Equation (41) was first discussed by Lord Rayleigh in 1883, in connection with vibrating systems subject to dissipating forces. As Rayleigh showed, no steady vibration is possible unless b_0 and b_2 have different signs.

Using the method of the first approximation we can show that, if c_1 is positive and the other parameters of (41) are small, the amplitude of the almost circular limit cycle in the $(z,\dot{z})$ phase plane is $2\sqrt{-b_0/b_2c_1}$. The amplitude of $\dot{z}$ $(= x)$ varies very little from this value with extremely high damping, the cycle becoming more elongated in the z direction (see also Briggs and Jones, 1953).

The physical reason for the occurrence of a limit cycle for systems satisfying either Rayleigh's equation or Van der Pol's equation can be seen by considering the energy function E defined by

$$E = \tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}c_1x^2 \ . \tag{44}$$

Differentiating (44) with respect to t, and using (31) we obtain

$$\frac{dE}{dt} = - f(x) \left(\frac{dx}{dt}\right)^2 \tag{45}$$

Now the total increase in energy in a cycle is zero, i.e. the energy supplied to the system balances that dissipated. From (45) we see that, for this to be possible, $f(x)$ must change sign as the system performs the cyclic oscillation.

1.4 Systems with damping and negative stiffness

Case 7 $\qquad b_2 = 0, \ c_1 < 0.$

$$\ddot{x} + b_0\dot{x} + c_1x = 0 \tag{46}$$

Equation (46) is a linear differential equation. The general solution and the trajectories in the phase plane were discussed in the previous paper (Babister, 1973).

Case 8 $\qquad b_2 > 0, \ c_1 < 0.$

$$\ddot{x} + (b_0+b_2x^2)\dot{x} + c_1x = 0 \tag{47}$$

Due to the change in the sign of c_1, the trajectories in the $(x,\dot{x})$ phase plane are very different from those of case 5. By examining the linear terms in (47) we find that the origin O is the only singular point and is a saddle point (this corresponds to an unstable position of equilibrium).

All the trajectories tend to infinity as $t \to \infty$, with the exception of the two trajectories (two separatrices of the singular point 0) which converge on the origin.

<u>Case 9</u> $b_2 < 0, \quad c_1 < 0.$

$$\ddot{x} + (b_0+b_2x^2)\dot{x} + c_1x = 0 \tag{48}$$

On putting $t = -T$, (48) becomes

$$\frac{d^2x}{dT^2} - (b_0+b_2x^2)\frac{dx}{dT} + c_1x = 0 \tag{49}$$

which is of the same form as (47). Thus the integral curves are found by applying a time scaling factor $\beta = -1$ to those of case 8. All the trajectories tend to infinity as $t \to \infty$, with the exception of the two arms of the singular point 0.

PART 2

Non-linear differential equations of the form

$$\ddot{x} + (b_0 + b_2x^2)\dot{x} + c_1x + c_3x^3 = 0$$

2.1 Introduction

We consider solutions of the equation

$$\ddot{x} + (b_0+b_2x^2)\dot{x} + c_1x + c_3x^3 = 0 \tag{50}$$

or the equivalent system

$$\left.\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -(b_0+b_2x^2)y - c_1x - c_3x^3, \end{aligned}\right\} \tag{51}$$

where b_0, b_2, c_1 and c_3 are real constants $(c_3 \neq 0)$. The point 0 is an isolated singular point for this system. As in part 1, we shall show how the nature of the solution depends upon the initial conditions $x = x_0$, $y = \dot{x}_0 = y_0$ at $t = 0$.

2.2 Systems with cubic-law stiffness $(c_1 = 0,\ c_3 > 0)$

Case 10 $\qquad b_2 = 0,\ c_1 = 0,\ c_3 > 0$

$$\ddot{x} + b_0\dot{x} + c_3x^3 = 0 \tag{52}$$

We first consider the equation

$$\ddot{x} + c_3x^3 = 0 \quad \text{with} \quad c_3 > 0. \tag{53}$$

The general first integral of (53) is

$$y^2 + \tfrac{1}{2}c_3x^4 = A \tag{54}$$

where $y = \dot{x}$ and A is a positive constant (or zero). The general solution for x can be expressed in terms of Jacobian elliptic functions. We find

$$x = (2A/c_3)^{1/4} \operatorname{cn}\left[(2Ac_3)^{1/4}\,(t+B),\ 1/\sqrt{2}\right] \tag{55}$$

where B is an arbitrary constant. The nature of the solution is most easily seen in the (x,y) phase plane (fig. 8) in which $y/\sqrt{c_3}$ is plotted against x, for $b_0=0$. We see that the trajectories are all closed curves (cycles) surrounding 0, which is a centre; there are, however, no limit cycles (this also follows since the system is conservative). From (55), the period is $4K(2Ac_3)^{-1/4}$, where K is the complete elliptic integral given by

$$K = F\left(\frac{\pi}{2}, \frac{1}{\sqrt{2}}\right) = 1.8541.$$

For the trajectory passing through the point $(x_0,0)$, the period is $4K/(c_3^{1/2}\,x_0)$. Thus the period is inversely proportional to the amplitudes in x.

If $b_0 > 0$, (52) has no general first integral. However, as in case 5 above, it can readily be shown that all the curves in the phase plane cross the curves of the family (54) in the inward direction, and approach the origin as a limit as $t \to \infty$. Every motion is damped out and no periodic motion exists. If $b_0 < 0$, all trajectories go to infinity as $t \to \infty$.

<u>Case 11</u> $\qquad b_2 > 0,\ c_1 = 0,\ c_3 > 0$

$$\ddot{x} + (b_0 + b_2x^2)\dot{x} + c_3x^3 = 0 \tag{56}$$

or the equivalent system

$$\left.\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -(b_0 + b_2x^2)y - c_3x^3. \end{aligned}\right\} \tag{57}$$

There is in general no first integral. However, in certain circumstances, a particular integral can be found. Thus, if $b_0b_2 = c_3$, (56) has the particular integral

$$\dot{x} = -b_0x. \tag{58}$$

The corresponding trajectories in the (x,y) phase plane are straight lines converging on 0 (since $b_0 > 0$). Putting $u = \dot{x} + b_0x$, we see that (56) becomes

$$\dot{u} + b_2ux^2 = 0. \tag{59}$$

It can be shown that no trajectory crosses the line $y = -b_0x$. Trajectories to the right of this line (in the first, second and fourth quadrants) turn in towards 0, having a common tangent along Ox at O. Similarly for trajectories to the left of this line.

Again, if $b_0 b_2 = 3c_3$, (56) has the particular integral

$$\dot{x} = -\frac{1}{3} b_2 x^3 , \tag{60}$$

or, in the phase plane, $y = -\frac{1}{3} b_2 x^3$. Putting $u = \dot{x} + \frac{1}{3} b_2 x^3$, we see that (56) becomes (with $b_0 b_2 = 3c_3$)

$$\dot{u} + b_0 u = 0,$$

which gives the first integral of (56) in the form

$$\dot{x} + \frac{1}{3} b_2 x^3 = C \exp(-b_0 t) \tag{61}$$

where C is an arbitrary constant. Thus, with $b_0 > 0$ and $b_0 b_2 = 3c_3$, the trajectories tend to merge with the cubic curve $y = -\frac{1}{3} b_2 x^3$ as t increases.

Equation (56) is a particular case of the more general equation

$$\ddot{x} + f(x)\dot{x} + c_3 x^3 = 0 \tag{62}$$

or the equivalent system

$$\left.\begin{aligned} \dot{x} &= y , \\ \dot{y} &= -f(x)y - c_3 x^3 \end{aligned}\right\} \tag{63}$$

The case in which $f(x)$ is identically zero was considered above (case 10), and it was shown that the curves in the phase plane were concentric quadrics, given by (54). By precisely similar reasoning to that of para. 1.3, it can be shown that, if $f(x)$ is everywhere positive (except, possibly, for $x = 0$), all the curves in the phase plane cross the curves of the family of quadrics (54) in the inward direction and approach the origin as a limit as $t \to \infty$. Thus, if in (56) $b_0 \geqslant 0$, every motion is damped out and no periodic motion exists. In particular, no limit cycles occur for the system (57) if b_0 and b_2 are both positive. This result also follows from Bendixson's theorem (Bendixson, 1901).

If $f(x)$ is not of invariable sign, periodic solutions and limit cycles may occur, the solution in the phase plane being bounded for $t > t_0$. As in case 5, if $b_0 < 0$ and $b_2 > 0$, (56) has a positive damping term if $|x|$ is large and a negative one if $|x|$ is small. On putting $x = \alpha X$, $t = \beta T$, where $\alpha = \sqrt{-b_0/b_2}$ and $\beta = \sqrt{-b_2/b_0 c_3} > 0$, (56) reduces to

$$\frac{d^2X}{dT^2} + \beta b_0 \,(1 - X^2)\, \frac{dX}{dT} + X^3 = 0. \qquad (64)$$

As shown by Levinson and Smith (1942), by means of a topological method, (64) has one and only one limit cycle, which is stable if $b_0 < 0$. This is shown, for equation

(56), in figure 9, in which $(-b_2y/b_0\sqrt{c_3})$ is plotted against $\sqrt{-b_2/b_0}\,x$ for $b_0\sqrt{-b_2/b_0c_3} = -1$. As in case 5, the limit cycle divides the phase plane into two regions. Trajectories which start near the origin spiral away from it ultimately becoming asymptotic to the limit cycle; those which start far from the origin also spiral in towards the same cycle.

The general shape of the limit cycle is similar to that for the Van der Pol equation; the maximum x amplitude differs only slightly from $2\sqrt{-b_0/b_2}$. For given values of b_0 and b_2, the maximum y amplitude is increased; this is to be expected since, for small values of x, the stiffness term c_3x^3 has much less effect on the slope of the limit cycle than the corresponding term c_1x in equation (50).

With zero damping, as in case 10, the period is $7.4/(x_0c_3^{1/2})$; if the damping is small, the period is approximately $3.7\sqrt{-b_2/b_0c_3}$. As with the Van der Pol equation, the period increases with the damping (as shown in figure 10).

<u>Case 12</u> $\quad b_2 < 0,\ c_1 = 0,\ c_3 > 0$

$$\ddot{x} + (b_0+b_2x^2)\dot{x} + c_3x^3 = 0 \qquad (65)$$

On putting $t = -T$, (65) becomes

$$\frac{d^2x}{dT^2} - (b_0+b_2x^2)\frac{dx}{dT} + c_3x^3 = 0. \tag{66}$$

Thus the integral curves can be found by applying a scaling factor $\beta = -1$ to those of case 11. If $b_0 \leqslant 0$, the trajectories tend to infinity as t increases. If $b_0 > 0$ and $b_2 < 0$, there is an unstable limit cycle. Trajectories which start inside the cycle spiral in towards the origin (which is a stable focus); those which start outside the cycle go off to infinity. Linear trajectories, corresponding to the particular integral (58), occur for $b_0b_2 = c_3$.

2.3 Systems with cubic-law stiffness ($c_1 = 0$, $c_3 < 0$)

Case 13 $\qquad b_2 = 0,\ c_1 = 0,\ c_3 < 0$

$$\ddot{x} + b_0\dot{x} + c_3x^3 = 0 \tag{67}$$

We first consider the equation

$$\ddot{x} + c_3x^3 = 0 \quad \text{with} \quad c_3 < 0 . \tag{68}$$

The general first integral of (68) is

$$y^2 + \tfrac{1}{2}c_3x^4 = A \tag{69}$$

where $y = \dot{x}$ and A is a constant. As in case 10, the general solution for x can be expressed in terms of Jacobian elliptic functions. We find

$$\frac{1}{x} = \mp \left(-\frac{c_3}{2A}\right)^{1/4} \quad \mathrm{tn}(u,\ 1/\sqrt{2}) \quad \mathrm{dn}(u,\ 1/\sqrt{2}), \quad (A > 0) \qquad (70)$$

where

$$u = (-\tfrac{1}{2}Ac_3)^{1/4}\ (t+B) \qquad (71)$$

and

$$\frac{1}{x} = \pm \left(\frac{c_3}{2A}\right)^{1/4} \quad \mathrm{cn}\left[(2Ac_3)^{1/4}\ (t+B),\ \ 1/\sqrt{2}\right], \quad (A < 0). \qquad (72)$$

In (70), the negative sign on the r.h.s. gives a solution for which y is always positive, and the positive sign gives one for which y is always negative. In (72), the positive sign on the r.h.s. gives a solution for which x is always positive and the negative sign gives one for which x is always negative. In (71) and (72), B is an arbitrary constant. The nature of the solutions is most easily seen in the (x, y) phase plane (figure 11). We see that in general the trajectories tend to infinity as t increases, except for the two arms of the separatrix $(A = 0)$ which are part of the parabolas $y = \pm \sqrt{-\tfrac{1}{2}c_3}\ x^2$.

If $b_0 \neq 0$, (67) has no general first integral. However, as in case 7 above, it can readily be shown that all the curves in the phase plane tend to infinity as $t \to \infty$, with the exception of the two trajectories (two separatrices of the singular point 0) which converge on the origin.

<u>Case 14</u> $\qquad b_2 > 0, \quad c_1 = 0, \quad c_3 < 0$

Case 15 $\qquad b_2 < 0, \quad c_1 = 0, \quad c_3 < 0$

$$\ddot{x} + (b_0+b_2x^2)\dot{x} + c_3x^3 = 0 \tag{73}$$

or the equivalent system

$$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= -(b_0+b_2x^2)y - c_3x^3 \end{aligned}\right\} \tag{74}$$

On putting $t = -T$, (73) becomes

$$\frac{d^2x}{dT^2} - (b_0+b_2x^2)\frac{dx}{dT} + c_3x^3 = 0 \tag{75}$$

Thus the integral curves for case 15 can be found by applying a scaling factor $\beta = -1$ to those of Case 14.

There is in general no first integral. However, as in Case 11 above, a particular integral can be found if $b_0b_2 = c_3$, or if $b_0b_2 = 3c_3$, the corresponding integrals being given by (58) and (60). It is readily shown that the origin is a saddle point of the system (74); all the curves in the phase plane tend to infinity as $t \to \pm\infty$, with the exception of the two trajectories (two separatrices of the singular point 0) which converge on the origin.

2.4 Systems with cubic stiffness $(c_1 > 0, \quad c_3 > 0)$

As pointed out in the general introduction, if both c_1 and c_3 are positive, the origin 0 is the only real singular point. The trajectories in the phase plane are similar to those discussed in paragraph 1.3, and, as we shall see below, limit cycles can occur.

Case 16 $\qquad b_2 = 0, \quad c_1 > 0, \quad c_3 > 0$

$$\ddot{x} + b_0\dot{x} + c_1 x + c_3 x^3 = 0 \tag{76}$$

We first consider the equation

$$\ddot{x} + c_1 x + c_3 x^3 = 0, \text{ with } c_1 > 0, \ c_3 > 0 \tag{77}$$

The general first integral of (77) is

$$y^2 + c_1 x^2 + \tfrac{1}{2}c_3 x^4 = A \tag{78}$$

where

$$y = \dot{x}$$

and A is a positive constant (or zero). As in paragraph 2.2, the general solution for x in terms of t can be expressed in terms of Jacobian elliptic functions.

We find

$$x = C \operatorname{cn}\left[\lambda\,(t+B),\ (c_3 C^2/2\lambda^2)^{\frac{1}{2}}\right] \tag{79}$$

where C is the positive root of

$$\tfrac{1}{2}c_3 C^4 + c_1 C^2 = A,$$

$$\lambda^2 = c_1 + c_3 C^2$$

and B is an arbitrary constant. We see that the general form of (79) is similar to that of (55). The trajectories in the phase plane are all closed curves (cycles) surrounding

0, which is a centre; however, there are no limit cycles. From (79), the period is $4K/\lambda$ where K is the complete elliptic integral given by

$$K = F\left[\frac{\pi}{2},\ (c_3C^2/2\lambda^2)^{\frac{1}{2}}\right].$$

If $b_0 \neq 0$, (76) has no general first integral. As in Case 10, it can readily be shown that, if $b_0 > 0$, all trajectories in the phase plane approach the origin as a limit as $t \to \infty$; if $b_0 < 0$, all trajectories go to infinity as $t \to \infty$.

Case 17 $\qquad b_2 > 0, \quad c_1 > 0, \quad c_3 > 0$

$$\ddot{x} + (b_0+b_2x^2)\dot{x} + c_1x + c_3x^3 = 0 \tag{80}$$

or the equivalent system

$$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= -(b_0+b_2x^2)y - c_1x - c_3x^3 \end{aligned}\right\} \tag{81}$$

This is the most general case of (50) considered so far. Equation (80) has no first integral; this equation is of the form

$$\ddot{x} + f(x)\dot{x} + g(x) = 0 \tag{82}$$

with $g(x)$ an odd function of x and $xg(x) > 0$ for all $x \neq 0$. The case $f(x) = 0$ has been dealt with in case 16. If $f(x) \neq 0$, the slope of the curves in the phase plane is given by

$$\frac{dy}{dx} = - f(x) - \frac{g(x)}{y} \quad (83)$$

From (83) we see that if $f(x)$ is everywhere positive (except, possibly, for $x = 0$), all the curves in the phase plane cross the simple closed curves of the family (78) in the inward direction and approach the origin as a limit as $t \to \infty$.

If $f(x)$ is not of invariable sign, and sustained oscillations may occur. Thus, if $b_0 < 0$ and $b_2 > 0$, as shown by Levinson and Smith (1942) by means of a topological method, the system (81) has one and only one limit cycle, which is stable. This closed curve surrounds the singular point 0, as shown for equation (80) in figure 12, for the values $b_0 = -1$, $b_2 = 1$, $c_1 = 1$, $c_3 = 1$. If $b_0 < 0$, 0 is an unstable focus. As in case 5, trajectories which start near the origin spiral away from it towards the limit cycle; those which start far from the origin also spiral in towards the same cycle.

<u>Case 18</u> $\qquad b_2 < 0, \quad c_1 > 0, \quad c_3 > 0$

$$\ddot{x} + (b_0+b_2x^2)\dot{x} + c_1x + c_3x^3 = 0 \quad (84)$$

On putting $t = - T$, (84) becomes

$$\frac{d^2x}{dT^2} - (b_0+b_2x^2)\frac{dx}{dT} + c_1x + c_3x^3 = 0 \quad (85)$$

which is of the same form as (80). The integral curves for this case are found by applying a time scaling factor $\beta = -1$ to those of case 17. If $b_0 \lessgtr 0$, with $b_2 < 0$, all the phase plane trajectories diverge to infinity as $t \to \infty$. If $b_0 > 0$ and $b_2 < 0$, there is one limit cycle, which is unstable (as in case 6).

2.5 Systems with cubic stiffness ($c_1 > 0$, $c_3 < 0$)

For these systems there are three singular points, at the origin 0 and at the points $x = \pm\sqrt{-c_1/c_3}$, these latter points both being saddle points (if $c_1 > 0$). As we shall show below, limit cycles can occur.

Case 19 $\quad b_2 = 0$, $c_1 >$, $c_3 < 0$

$$\ddot{x} + b_0\dot{x} + c_1 x + c_3 x^3 = 0 \tag{86}$$

We first consider the equation

$$\ddot{x} + c_1 x + c_3 x^3 = 0, \text{ with } c_1 > 0,\ c_3 < 0. \tag{87}$$

The general first integral of (87) is

$$y^2 + c_1 x^2 + \tfrac{1}{2}c_3 x^4 = A \tag{88}$$

where

$$y = \dot{x}$$

and A is a constant. As in paragraph 2.2, the general solution for x in terms of t can be expressed in terms of Jacobian elliptic functions. We find

$$\frac{1}{x} = \mp \left(-\frac{c_3}{2A}\right)^{1/4} \quad \mathrm{tn}\,(u,k)\ \mathrm{dn}\,(u,k)\ , \quad (A \geqslant -c_1{}^2/2c_3) \qquad (89)$$

where

$$u = (-\tfrac{1}{2}Ac_3)^{1/4} \quad (t+B)$$

and

$$k^2 = \tfrac{1}{2} + \frac{c_1}{2\sqrt{-2Ac_3}}\ ,$$

$$x = C\ \mathrm{sn}\left[\lambda\ (t+B),\ (-c_3C^2/2\lambda^2)^{\frac{1}{2}}\right],\ (-c_1{}^2/2c_3 \geqslant A > 0) \qquad (90)$$

where C is the smaller positive root of

$$\tfrac{1}{2}\ c_3C^4 + c_1C^2 = A$$

and

$$\lambda^2 = c_1 + \tfrac{1}{2}\ c_3C^2,$$

$$x = \pm\ C/\mathrm{sn}\left[(-\tfrac{1}{2}c_3C^2)^{\frac{1}{2}}\ (t+B),\ k\right]\ ,\ (-c_1{}^2/2c_3 \geqslant A > 0) \qquad (91)$$

where C is the larger positive root of

$$\tfrac{1}{2}\ c_3C^4 + c_1C^2 = A$$

and

$$k^2 = -1 - 2\ (c_1/c_3C^2),$$

$$x = \pm\ C/\ \mathrm{cn}\left[\lambda\ (t+B),\ (-A/C^2\lambda^2)^{\frac{1}{2}}\right],\ (A < 0) \qquad (92)$$

where C is the positive root of

$$\tfrac{1}{2}c_3C^4 + c_1C^2 = A$$

and

$$\lambda^2 = -c_1 - c_3C^2.$$

In equations (89) - (92), B is an arbitrary constant. The phase plane solutions are given in figure 13. We see that there is a region surrounding 0 for which cyclic trajectories are possible. This region is bounded by the separatrices PAQ, QBP (which represent trajectories commencing at one saddle point and ending at the other). All trajectories outside this region go off to infinity as t increases (except the two arms of the separatrices which terminate at the saddle points).

If $b_0 \neq 0$, (86) has no general first integral. If $b_0 > 0$, there is a stable focus at 0. As shown in figure 14, the trajectories are of two types. Those within the region PAQ, P'BQ' (bounded by the separatrices of the saddle points at $x = \pm\sqrt{-c_1/c_3})$ converge on 0. All trajectories outside this region go off to infinity as t increases. If $b_0 < 0$, 0 is an unstable focus, and all trajectories go off to infinity as t increases.

<u>Case 20</u> $b_2 > 0, \quad c_1 > 0, \quad c_3 < 0$

$$\ddot{x} + (b_0 + b_2 x^2)\dot{x} + c_1 x + c_3 x^3 = 0 \tag{93}$$

This case has many similarities with that of the previous one (with damping present). Equation (93) has no general first integral. If $b_0 \geqslant 0$ the origin 0 is a stable focus, and (as shown in figure 15, in which $b_0 = 0$, $b_2 = 1$, $c_1 = 1$, $c_3 = -1$) trajectories within the region PAQ, P'BQ' converge on 0. If $b_0 < 0$, 0 is an unstable focus. From

(93) we see that, in this case, the system has negative damping for small values of $|x|$ and positive damping for large $|x|$. Now, for small values of c_3, the system (93) can be considered as a perturbed Van der Pol equation, and, by the theory of analytic continuation (Urabe, 1967), the perturbed system will have a unique limit cycle, which is stable if $b_0 < 0$ and $b_2 > 0$, and which lies in the neighbourhood of that of Van der Pol's equation.

From the analogue computer results it was found that limit cycles could occur for $(b_0 c_3/b_2 c_1) < 1/5$. For larger negative values of c_3, all the trajectories went off to infinity as t increased.

Case 21 $\quad b_2 < 0, \quad c_1 > 0, \quad c_3 < 0$

The integral curves for this case are found by applying a time scaling factor $\beta = -1$ to those of case 20. If $b_0 > 0$ limit cycles occur, provided that $(b_0 c_3/b_2 c_1) < 1/5$.

However, in this case, the limit cycles are unstable (0 is a stable focus if $b_0 > 0$).

2.6 Systems with cubic stiffness $(c_1 < 0, \quad c_3 > 0)$

For these systems, there are three singular points, at the origin 0 (a saddle point) and at the points $x = \pm\sqrt{-c_1/c_3}$. As shown below, limit cycles can occur.

Case 22 $\qquad b_2 = 0, \quad c_1 < 0, \quad c_3 > 0$

$$\ddot{x} + b_0\dot{x} + c_1 x + c_3 x^3 = 0 \tag{94}$$

We first consider the equation

$$\ddot{x} + c_1 x + c_3 x^3 = 0 \quad \text{with} \quad c_1 < 0, \; c_3 > 0 \tag{95}$$

The general first integral of (95) is

$$y^2 + c_1 x^2 + \tfrac{1}{2} c_3 x^4 = A$$

where $y = \dot{x}$ and A is a constant. As in paragraph 2.5, the general solution for x in terms of t can be expressed in terms of Jacobian elliptic functions.

We find

$$x = C \operatorname{cn}\left[\lambda\ (t+B),\ (c_3 C^2/2\lambda^2)^{\frac{1}{2}}\right] \quad A > 0 \tag{96}$$

where C is the positive root of

$$\tfrac{1}{2} c_3 C^4 + c_1 C^2 = A$$

and

$$\lambda^2 = c_1 + c_3 C^2 ,$$

$$x = (-2c_1/c_3)^{\frac{1}{2}} \operatorname{sech}\left[(-c_1)^{\frac{1}{2}}\ (t+B)\right] , \quad A = 0 \tag{97}$$

$$x = C \operatorname{dn}\left[(\tfrac{1}{2}c_3 C^2)^{\frac{1}{2}}\ (t+B),\ k\right], \quad -(c_1{}^2/2c_3) < A < 0 \tag{98}$$

where C is one of the positive roots of

$$\tfrac{1}{2} c_3 C^4 + c_1 C^2 = A$$

and

$$k^2 = 2 + 2(c_1/c_3 C^2).$$

In equations (96) - (98), B is an arbitrary constant.

The phase plane solutions are given in figure 16. We see that, if $b_0 = 0$, there are outer cyclic trajectories for $A > 0$ which enclose the origin and the two centres at $\pm\sqrt{-c_1/c_3}$; for $(-c_1^2/2c_3) < A < 0$, there are inner cyclic trajectories which enclose one of the centres. The figure 8 curve OPOQO, for which $A = 0$, is a separatrix for the system (it represents trajectories which start and finish at the saddle point 0).

If $b_0 \neq 0$, (94) has no general first integral. If $b_0 > 0$, there are stable foci at $x = \pm\sqrt{-c_1/c_3}$, to which all trajectories tend as $t \to \infty$ (except for the two trajectories which terminate at the saddle point 0); if $b_0 < 0$, there are unstable foci at $x = \pm\sqrt{-c_1/c_3}$; all the trajectories go to infinity as $t \to \infty$ (except for the two trajectories which terminate at the saddle point 0).

<u>Case 23</u> $\quad b_2 > 0, \quad c_1 < 0, \quad c_3 > 0$

$$\ddot{x} + (b_0+b_2x^2)\dot{x} + c_1x + c_3x^3 = 0 \tag{99}$$

or the equivalent system

$$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= -(b_0+b_2x^2)y - c_1x - c_3x^3. \end{aligned}\right\} \tag{100}$$

There is in general no first integral. However, in certain circumstances, a particular integral can be found. Thus (99) has the particular integral

$$b_2\dot{x} = - c_3 x \tag{101}$$

if

$$c_3^{\,2} - b_0 b_2 c_3 + b_2^{\,2} c_1 = 0 \tag{102}$$

The corresponding trajectories in the (x,y) phase plane are straight lines, converging on 0 (since $b_2/c_3 > 0$).

More generally, if $b_0 \geqslant 0$ with $b_2 > 0$, the trajectories converge on the stable foci (or nodes) at the points $x = \pm \sqrt{-c_1/c_3}$ (apart from the two trajectories which terminate at the saddle point 0). If $b_0 < 0$, the foci (or nodes) may become unstable. From (99) we see that, in this case, the system has negative damping for small values of $|x|$ and positive damping for large $|x|$. Now, for small values of c_1, the system (99) can be considered as a perturbed form of equation (56). By the theory of analytic continuation (as for case 20), we find that, if $c_3 > 0$, the system (100) with $b_0 < 0$ will have a stable limit cycle for small values of c_1. An upper bound to the values of $(-c_1)$ for which limit cycles occur is established by noting that, as shown above, (100) has linear trajectories passing through the origin if equation (102) is satisfied. This is the curve FOG shown in figure 17, in which the coordinates are $c_1/b_0^{\,2}$ and $c_3/(-b_0 b_2)$. It can be shown, from (100), that if x has the dimensions of length, these coordinates are non-dimensional, and the equation of any closed trajectory can be written in the form

$$\sqrt{-\frac{b_2}{b_0}}\, y = b_0\, f\left(\sqrt{-\frac{b_2}{b_0}}\, x,\ \frac{c_1}{b_0^2},\ \frac{c_3}{b_0 b_2}\right). \tag{103}$$

From the analogue computer results it was found that limit cycles could occur throughout the region of the (c_1, c_3) plane to the right of the curves OC and OE (figure 17), but not to the left of these curves. If the equations of these curves are written in the form

$$c_3 = \mu\,(b_2 c_1/b_0), \tag{104}$$

the variation of μ with $c_1/b_0^{\,2}$ is shown in figure 18. It can be seen that μ is practically constant for the curve OC and for all values of $|\, c_1/b_0^{\,2}\,|$ greater than unity on the curve OE. Thus OC and OE are linear boundaries for most of their length, for the range of values investigated, their slopes being approximately $-1/5$ and $-3/4$.

In figure 17, OD is the line $b_0 c_3 = b_2 c_1$. The system (100) with $b_0 < 0$, has two unstable foci (or nodes) for values of c_1 and c_3 in the region BOD. For such values of c_1 and c_3 there is one stable limit cycle; this is shown in figure 19 for $b_0 = -1$, $b_2 = 1$, $c_1 = -1$ and $c_3 = 1$, together with the corresponding curve for $c_1 = 0$. For this range of c_1 the maximum x amplitude of the limit cycle differs only slightly from $2\sqrt{-b_0/b_2}$. Figure 20 shows a trajectory in the phase plane for $b_0 = -1$, $b_2 = 1$, $c_1 = -1$, $c_3 = 1$. In this case the corresponding point in figure 17 lies on OD and the system has weak unstable foci.

For values of c_1 and c_3 in the region DOH, the system (100) with $b_0 < 0$ has two stable foci (or nodes) at $x = \pm \sqrt{-c_1/c_3}$. If the corresponding points in figure 17 lie in the region DOE, there is one outer stable limit cycle and also an inner unstable cycle surrounding each foci. Figure 21 shows the trajectories in the phase plane for $b_0 = -1$, $b_2 = 1$, $c_1 = -1$, $c_3 = 0.9$. For initial conditions $x = -1.4$, $\dot{x} = 0$ the trajectory spirals round and converges on the focus at $x = -1.05$, whereas for the initial conditions $x = -1.5$, $\dot{x} = 0$, the trajectory tends towards the outer limit cycle. The unstable limit cycle lies between these two trajectories.

In the region DOE, as c_3 decreases (for a given negative value of c_1), the outer stable limit cycle has an increasingly slender waist (and its period increases) until, as c_3 passes through the critical value (corresponding to points on the line OE), the outer and inner limit cycles merge to become a stable - unstable curve, passing through the saddle point at the origin. This is comparable with the case of the undamped system (b_0 and b_2 zero) considered above (case 22). For further decrease of c_3 (in the region EOH) no limit cycles occur, all the trajectories converging on one or other of the foci.

Case 24 $\qquad b_2 < 0, \quad c_1 < 0, \quad c_3 > 0$

The integral curves for this case are found by applying a time scaling factor $\beta = -1$ to those of case 23. If $b_0 > 0$ and $b_2 < 0$, the stable cycles of case 23 become unstable and vice versa. Thus in this case the system has two inner stable limit cycles for values of c_1 and c_3 in the region DOE in figure 17, together with an outer unstable cycle. These stable limit cycles enclose the corresponding foci $\pm\sqrt{-c_1/c_3}$, as shown in figure 22. As these foci become more unstable (for points near OE), the inner and outer limit cycles become nearer to one another and to the origin, finally merging as stated above.

2.7 Systems with cubic stiffness $(c_1 < 0, \quad c_3 < 0)$

For these systems, the origin 0 is the only real singularity and is a saddle point.

Case 25 $\qquad b_2 = 0, \quad c_1 < 0, \quad c_3 < 0$

$$\ddot{x} + b_0\dot{x} + c_1 x + c_3 x^3 = 0 \tag{105}$$

We first consider the equation

$$\ddot{x} + c_1 x + c_3 x^3 = 0 \quad \text{with} \quad c_1 < 0, \quad c_3 < 0 . \tag{106}$$

The general first integral of (106) is

$$y^2 + c_1 x^2 + \tfrac{1}{2} c_3 x^4 = A \tag{107}$$

where

$$y = \dot{x}$$

and A is a constant. As in paragraph 2.3, the general solution for x in terms of t can be expressed in terms of Jacobian elliptic functions. We find

$$\frac{1}{x} = \mp \left(-\frac{c_3}{2A}\right)^{1/4} \operatorname{tn}(u,k) \operatorname{dn}(u,k), \quad (A > 0) \qquad (108)$$

where

$$u = (-\tfrac{1}{2} Ac_3)^{1/4} (t+B)$$

and

$$k^2 = \tfrac{1}{2} + \frac{c_1}{2(-2Ac_3)^{1/2}} ,$$

$$x = \pm C/\operatorname{cn}\left[\lambda (t+B), (-A/C^2 \lambda^2)^{1/2}\right], \quad (A < 0) \qquad (109)$$

where C is the positive root of

$$\tfrac{1}{2}c_3 C^4 + c_1 C^2 = A$$

and

$$\lambda^2 = -c_1 - c_3 C^2$$

In (108) and (109), B is an arbitrary constant.

The form of these solutions is identical with that of (89) and (92). However, unlike case 19, there is no inner cyclic region. The nature of the solutions is most easily seen in the (x,y) phase plane (figure 23). As in case 13, in general the trajectories tend to infinity as t increases, except for the two arms of the separatrix $(A = 0)$ which are parts of the curves

$$y = \pm \sqrt{-c_1 x^2 - \tfrac{1}{2} c_3 x^4} .$$

If $b_0 \neq 0$, (105) has no general first integral. As in case 7, it can readily be shown that all the curves in the phase plane tend to infinity as $t \to \infty$, with the exception of the two trajectories (two separatrices of the singular point 0) which converge on the origin.

<u>Case 26</u> $b_2 > 0$, $c_1 < 0$, $c_3 < 0$

<u>Case 27</u> $b_2 < 0$, $c_1 < 0$, $c_3 < 0$

$$\ddot{x} + (b_0+b_2x^2)\dot{x} + c_1x + c_3x^3 = 0 \tag{110}$$

or the equivalent system

$$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= -(b_0+b_2x^2)y - c_1x - c_3x^3 \end{aligned}\right\} \tag{111}$$

On putting $t = -T$, (110) becomes

$$\frac{d^2x}{dT^2} - (b_0 + b_2x^2)\frac{dx}{dT} + c_1x + c_3x^3 = 0 \tag{112}$$

Thus the integral curves for case 27 can be found by applying a scaling factor $\beta = -1$ to those of case 26.

There is in general no first integral. However, as in case 22, a particular integral $b_2\dot{x} + c_3x = 0$ can be found if

$$c_3^2 - b_0 b_2 c_3 + b_2^2 c_1 = 0$$

It is readily shown that the origin is a saddle point of the system (111); all the curves in the phase plane tend to infinity as $t \to \pm \infty$, with the exception of the two trajectories (two separatrices of the singular point 0) which converge on the origin.

REFERENCES

Babister, A.W. Some results relating to certain general types of non-linear second order differential equation. University of Glasgow, Department of Aeronautics and Fluid Mechanics, Report 7201 (1972).

Babister, A.W. Non-linear differential equations having quadratic stiffness terms.
University of Glasgow, Department of Aeronautics and Fluid Mechanics, Report 7301 (1973).

Bendixson, I. Sur les courbes définies par des équations différentiells. Acta Math., 24 (1901), 1 - 88.

Briggs, B.R. and Jones, A.L. Techniques for calculating parameters of non-linear dynamic systems from response data.
NACA Tech Note 2977 (1953).

Lefschetz, S. Differential equations. Geometric theory. (Interscience, 1957).

Levinson, N. and Smith, O.K. A general equation for relaxation oscillations.
Duke Math. Jour., 9 (1942), 382 - 403.

Loud, W.S. Behaviour of the period of solutions of certain plane autonomous systems near centres.
Contributions to Differential Equations (Interscience, 1964).

Murphy, C.H. The effect of strongly non-linear static moment on the combined pitching and yawing motion of a symmetric missile. BRL Report, 1114, Aberdeen Proving Ground (1960).

Rayleigh, Lord. On maintained vibrations. Phil. Mag. 15 (1), (1883), 229.

Sansone, G. and Conti, R. Non-linear differential equations. (Pergamon, 1964).

Shinbrot, R. On the analysis of linear and non-linear dynamical systems from transient response data. NACA Tech Note 3288 (1954).

Urabe, M. Non-linear autonomous oscillations. (Academic Press, 1967).

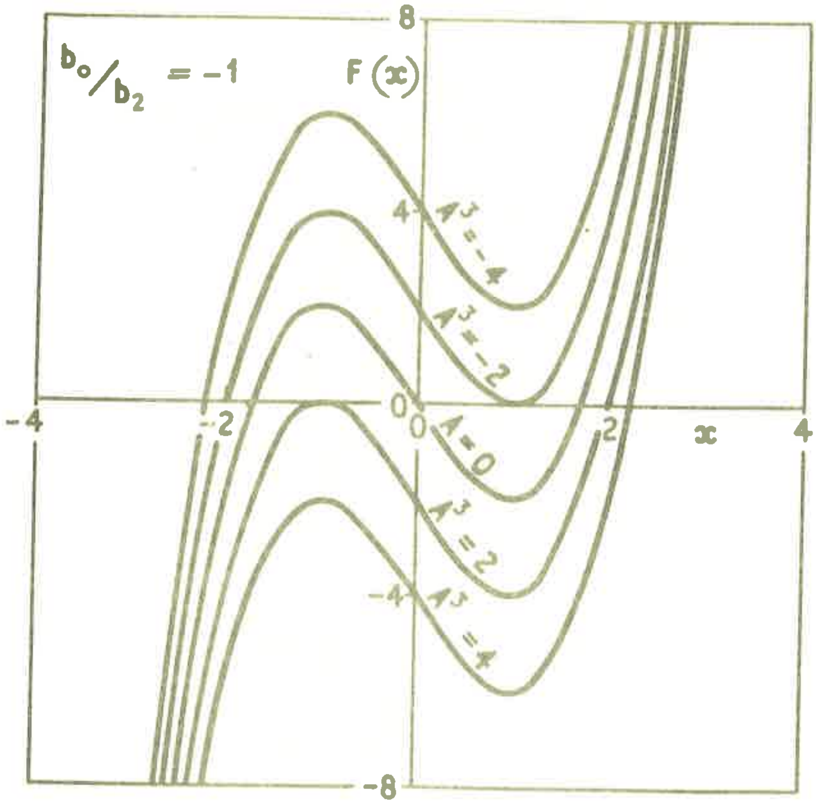

Fig. 1. VARIATION OF $F(x)$ WITH x

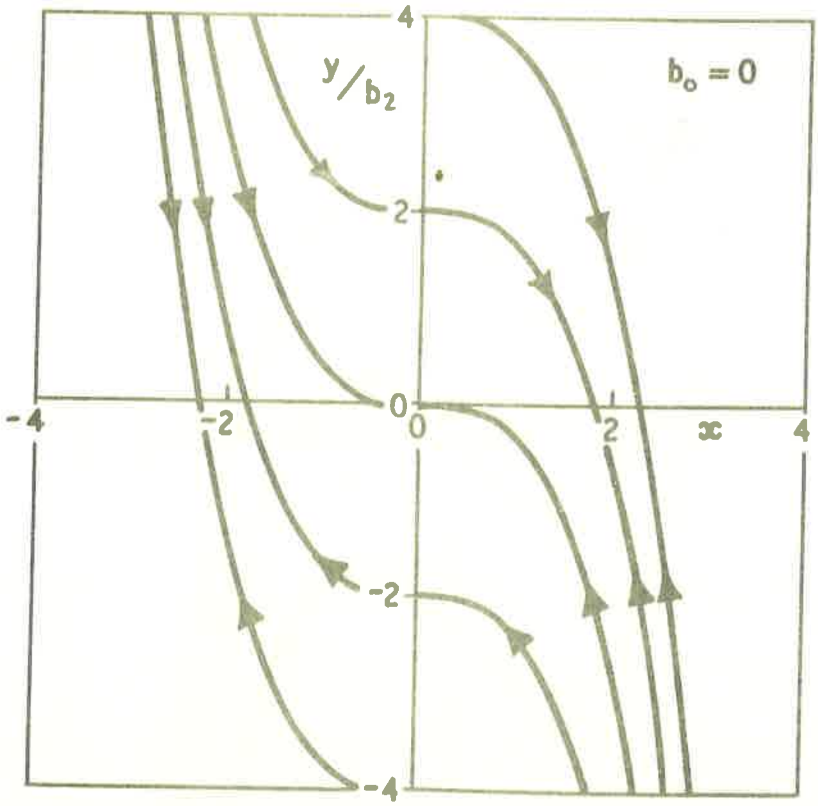

Fig. 2. TRAJECTORIES $b_2 > 0$, $c_1 = 0$, $c_3 = 0$

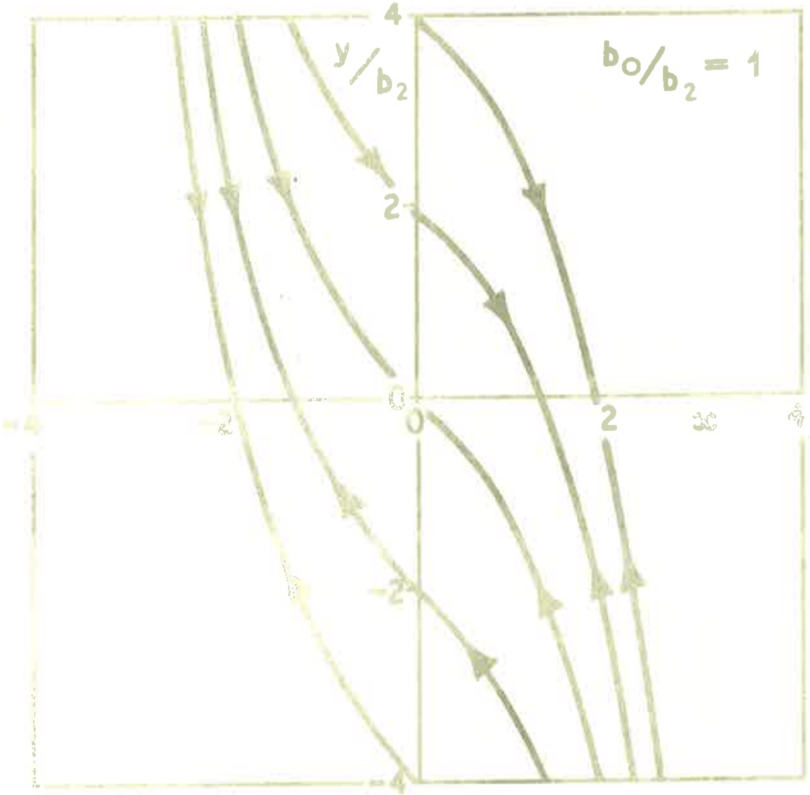

Fig. 3. TRAJECTORIES $b_2 > 0$, $c_1 = 0$, $c_3 = 0$

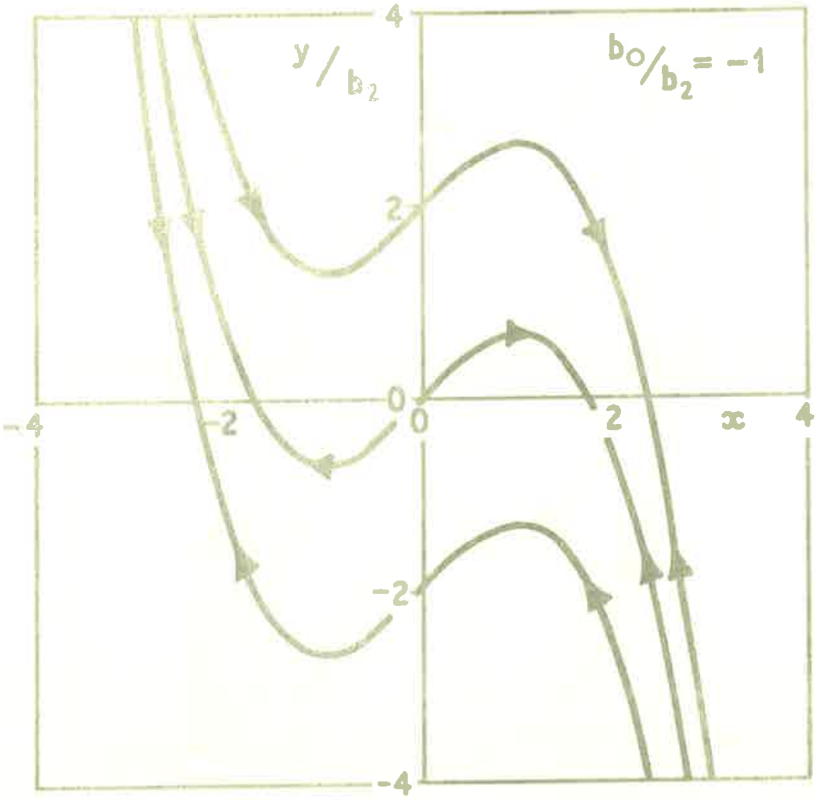

Fig. 4. TRAJECTORIES $b_2 > 0$, $c_1 = 0$, $c_3 = 0$

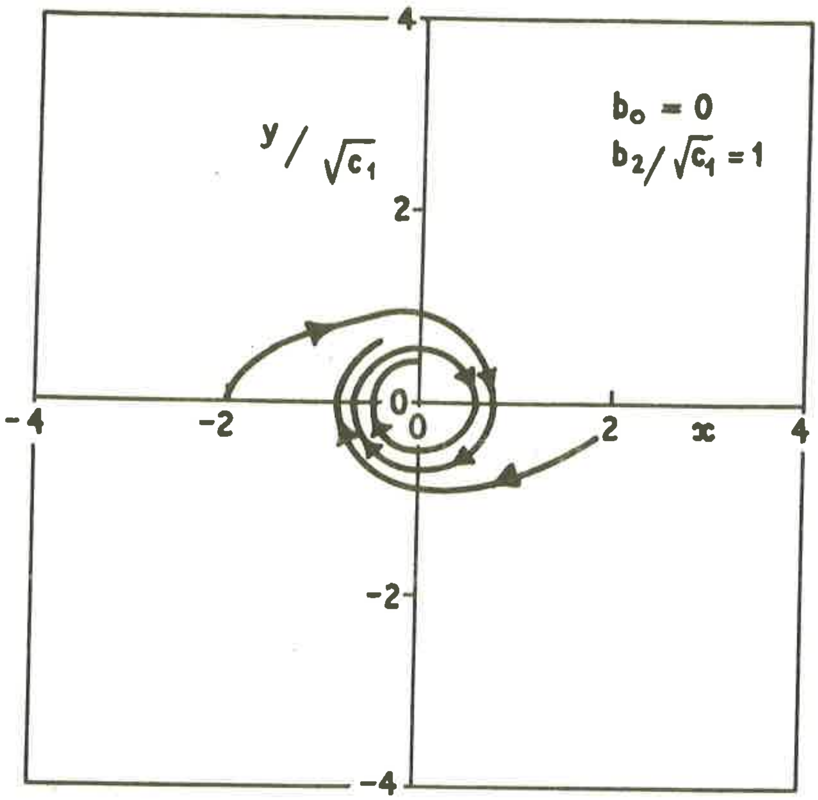

Fig. 5. TRAJECTORIES $b_2 > 0, c_1 > 0, c_3 = 0$

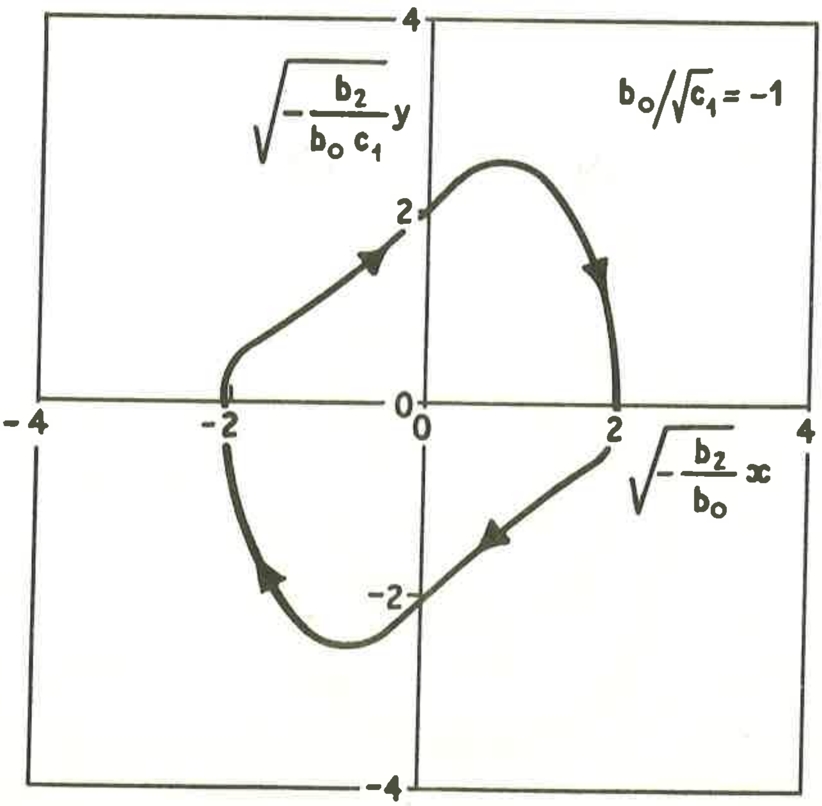

Fig. 6. LIMIT CYCLE $b_2 > 0 , c_1 > 0, c_3 = 0$

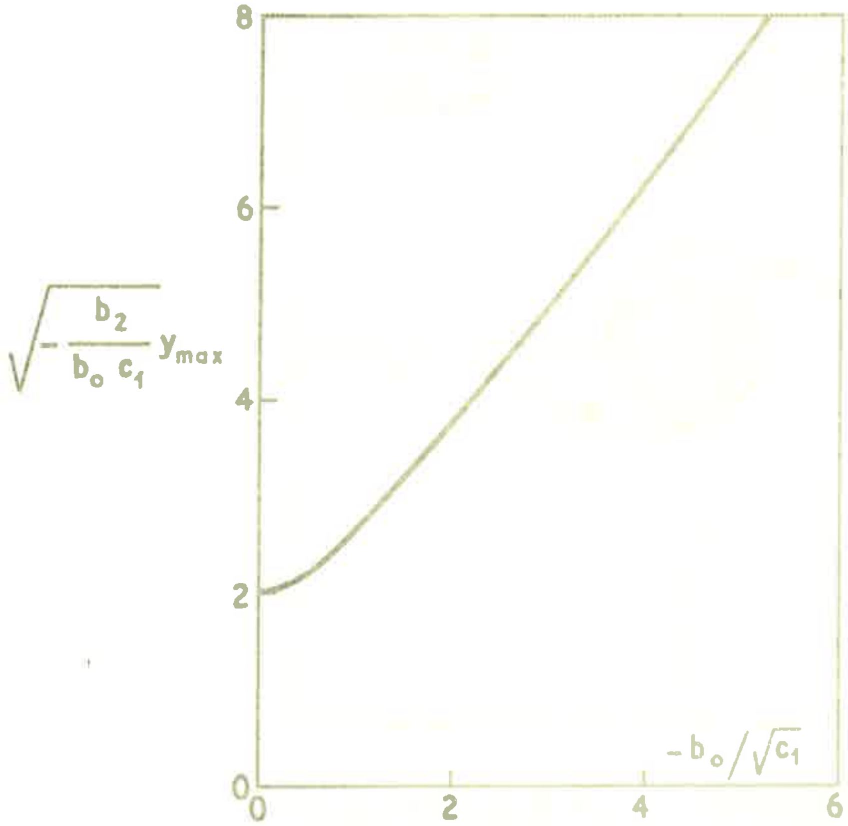

Fig. 7. y AMPLITUDE OF LIMIT CYCLE

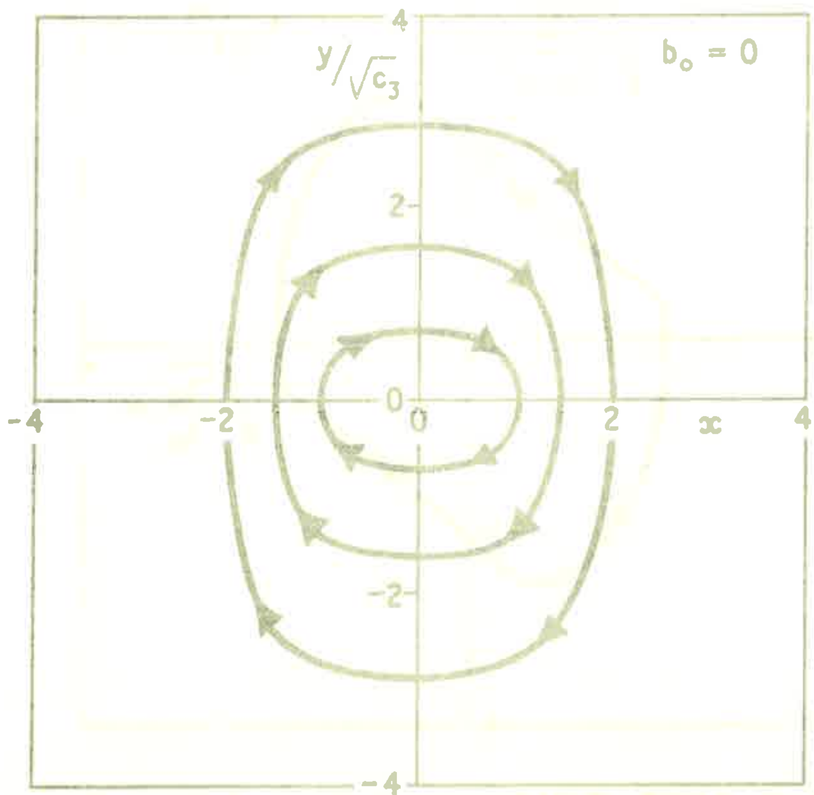

Fig. 8. TRAJECTORIES $b_2 = 0$, $c_1 = 0$, $c_3 > 0$

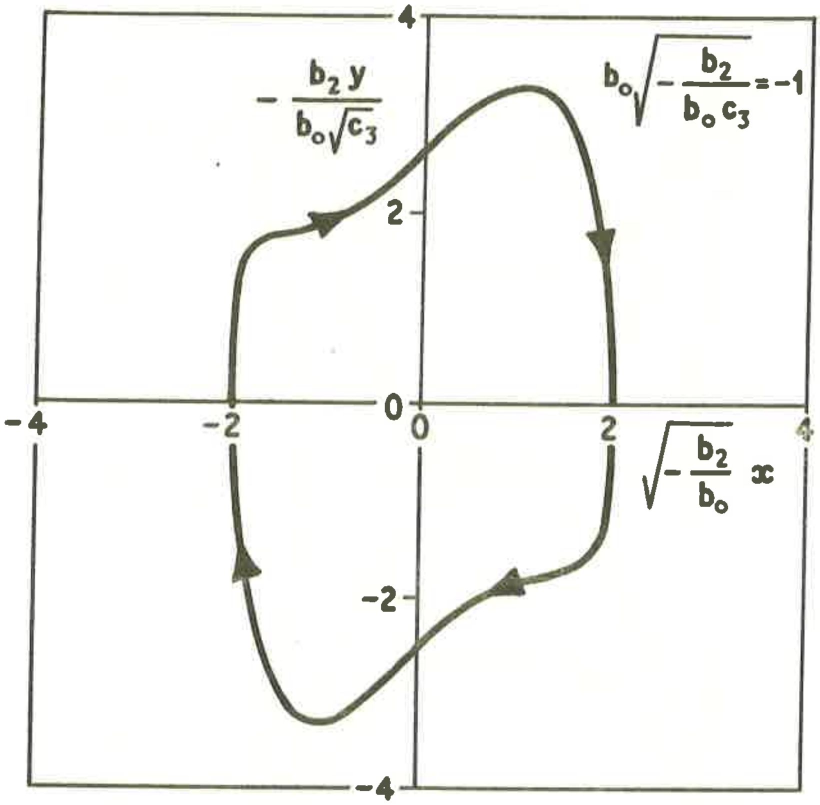

Fig. 9. LIMIT CYCLE $b_2 > 0$, $c_1 = 0$, $c_3 > 0$

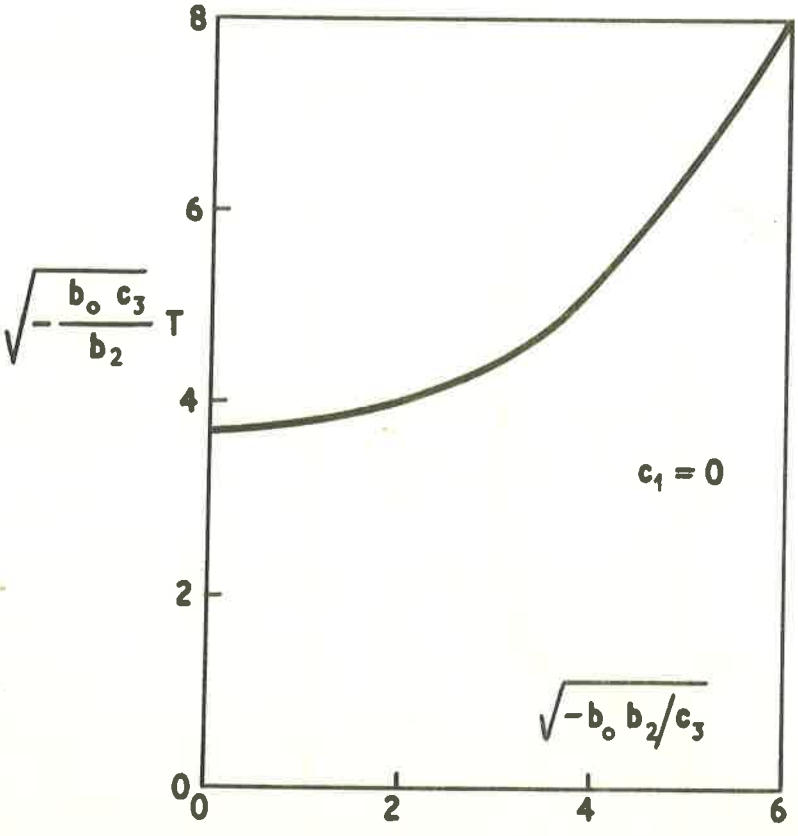

Fig. 10. PERIOD T OF LIMIT CYCLE

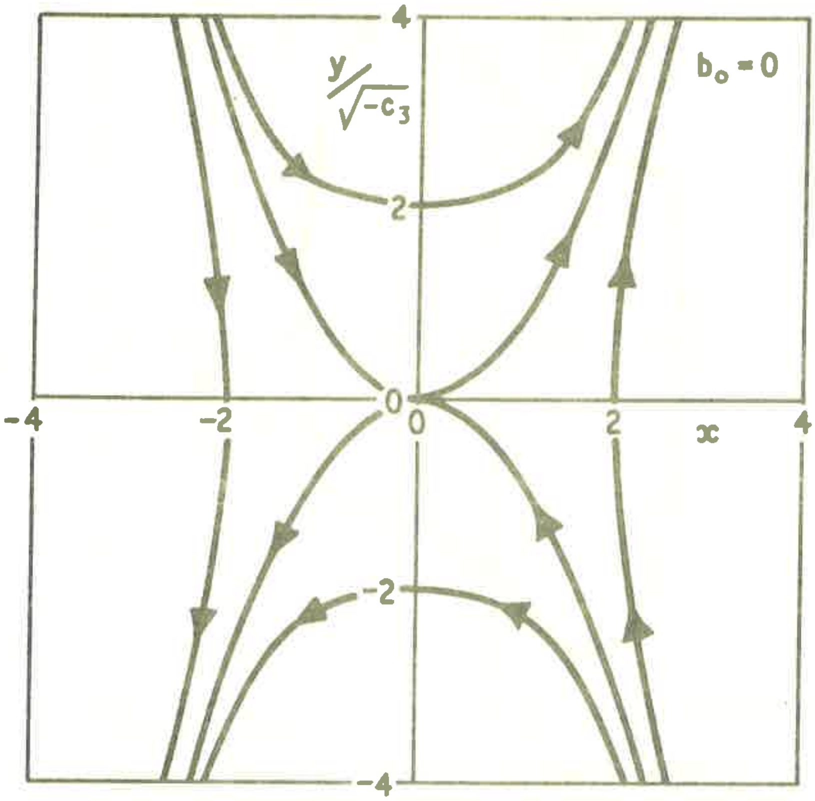

Fig. 11. TRAJECTORIES $b_2 = 0$, $c_1 = 0$, $c_3 < 0$

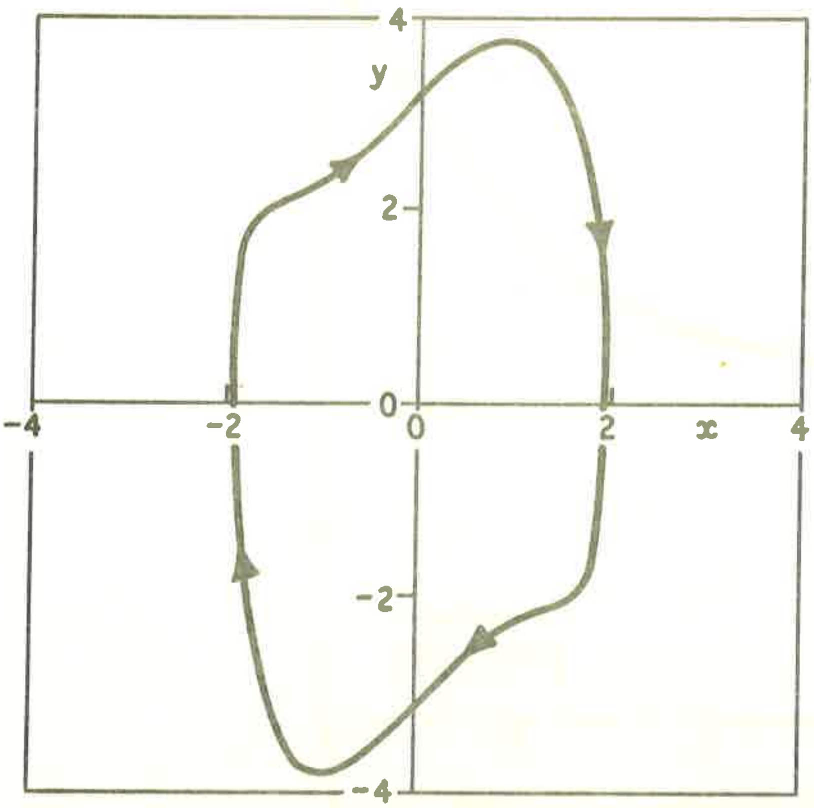

Fig. 12. LIMIT CYCLE $b_0 = -1$, $b_2 = 1$, $c_1 = 1$, $c_3 = 1$

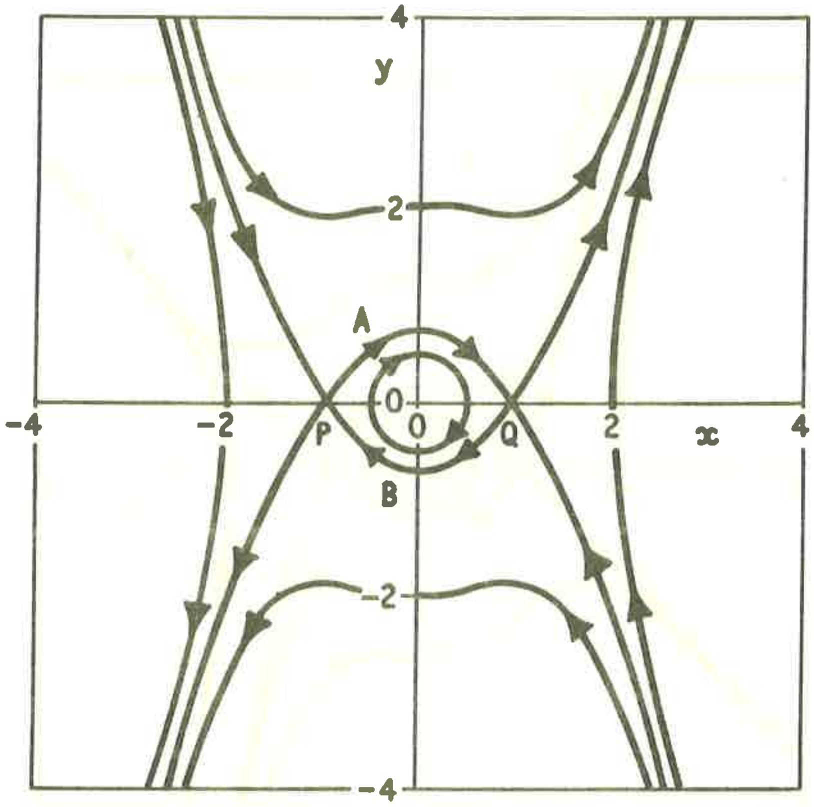

Fig. 13. TRAJECTORIES $b_0 = 0, b_2 = 0, c_1 = 1, c_3 = -1$

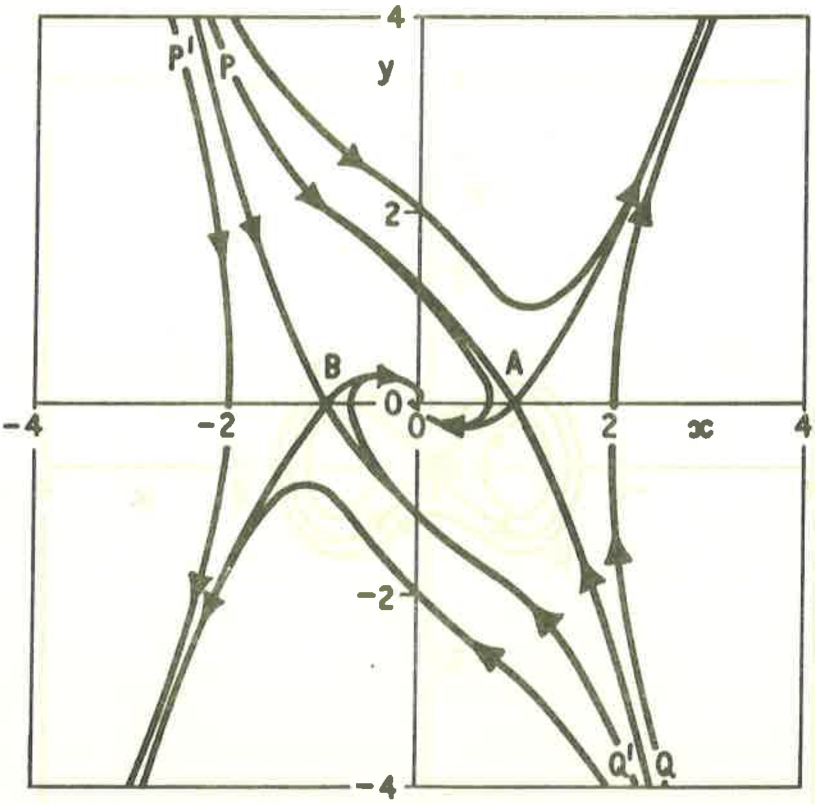

Fig. 14. TRAJECTORIES $b_0 = 1, b_2 = 0, c_1 = 1, c_3 = -1$

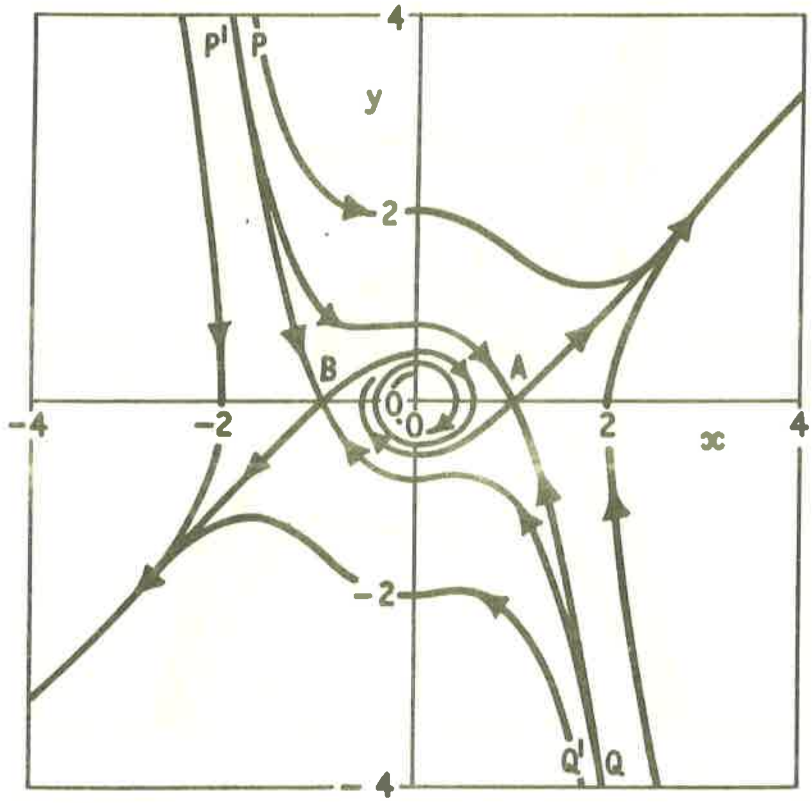

Fig. 15. TRAJECTORIES $b_0 = 0, b_2 = 1, c_1 = 1, c_3 = -1$

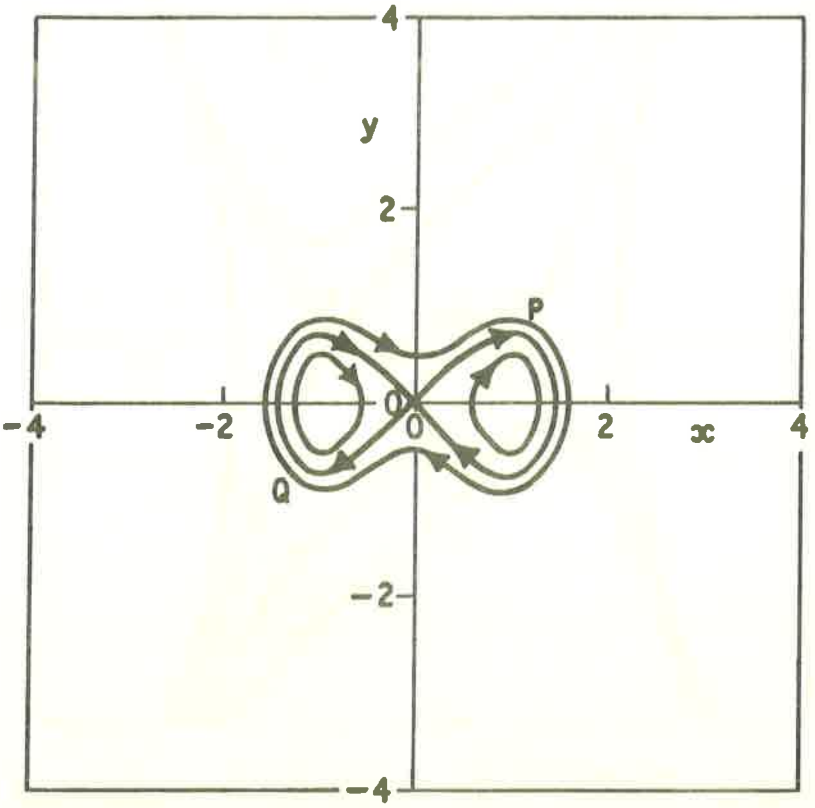

Fig. 16. TRAJECTORIES $b_0 = 0, b_2 = 0, c_1 = -1, c_3 = 1$

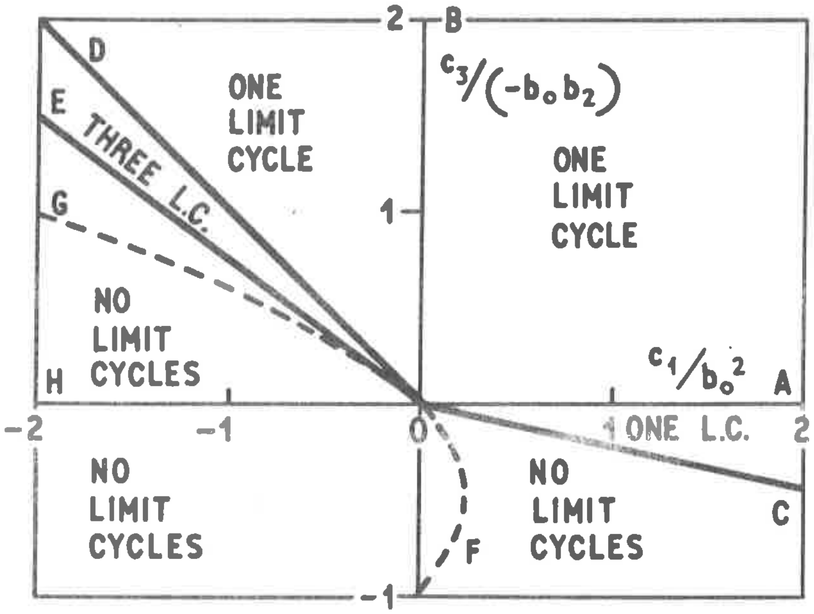

Fig. 17. OCCURRENCE OF LIMIT CYCLES

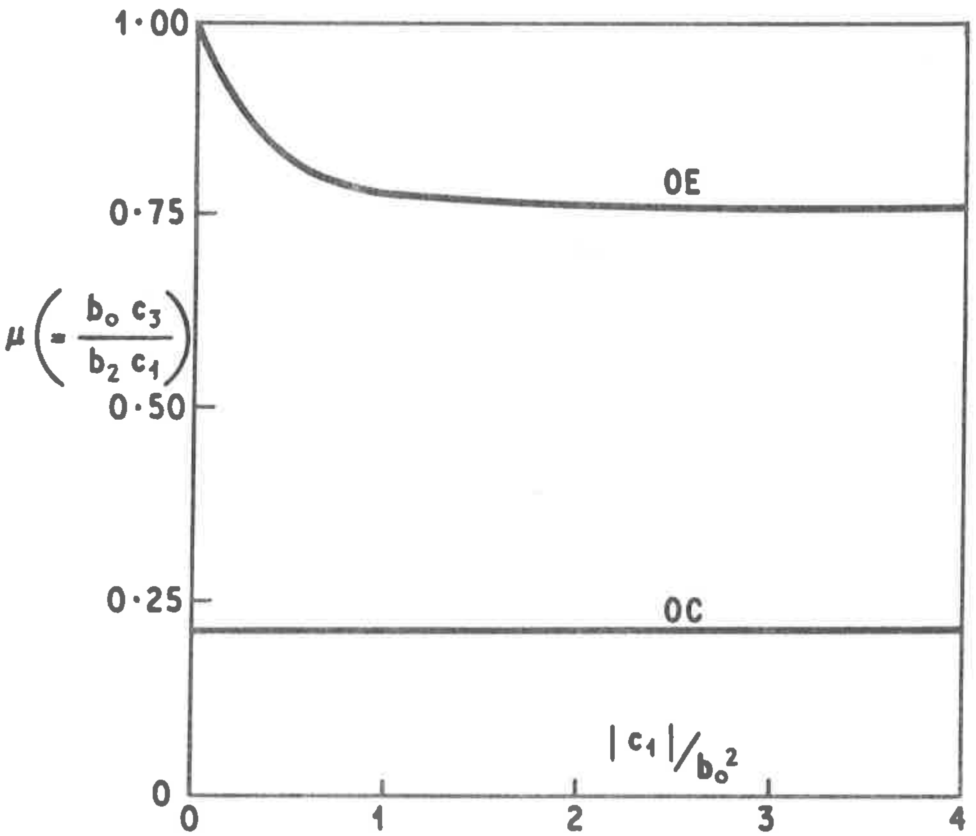

Fig. 18. CRITICAL VALUES OF THE PARAMETERS

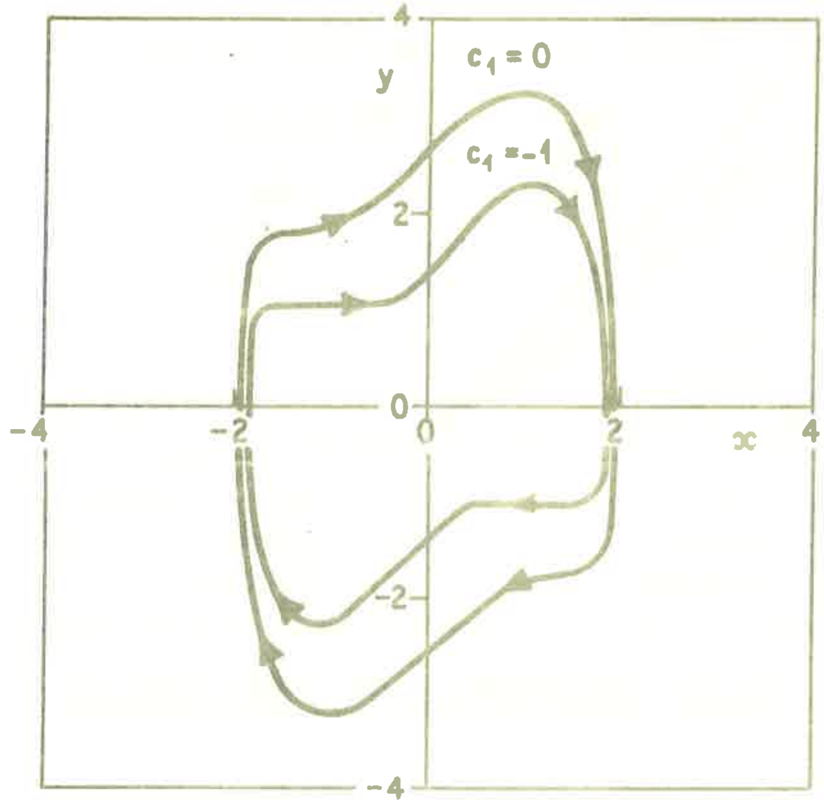

Fig. 19. LIMIT CYCLES $b_0 = -1$, $b_2 = 1$, $c_3 = 1$

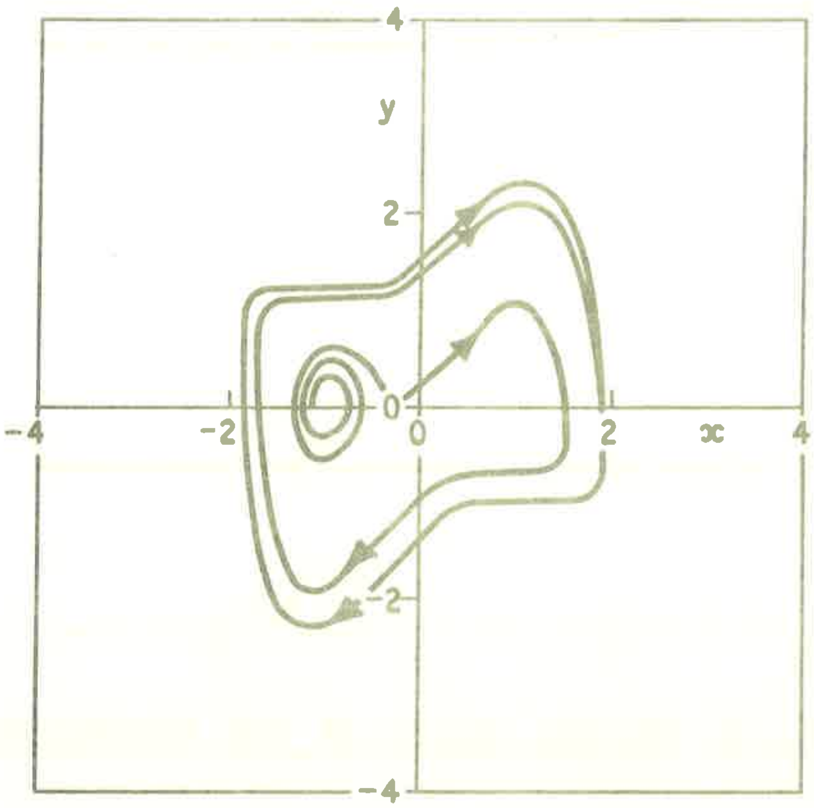

Fig. 20. TRAJECTORY $b_0 = -1, b_2 = 1, c_1 = -1, c_3 = 1$

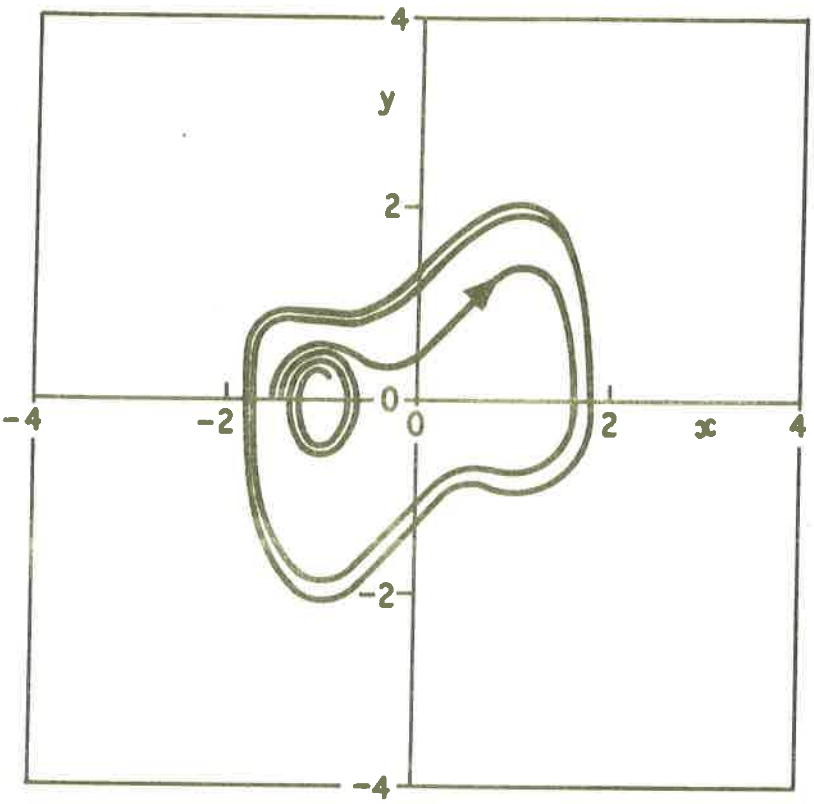

Fig. 21. TRAJECTORIES $b_0=-1, b_2=1, c_1=-1, c_3=0{\cdot}9$

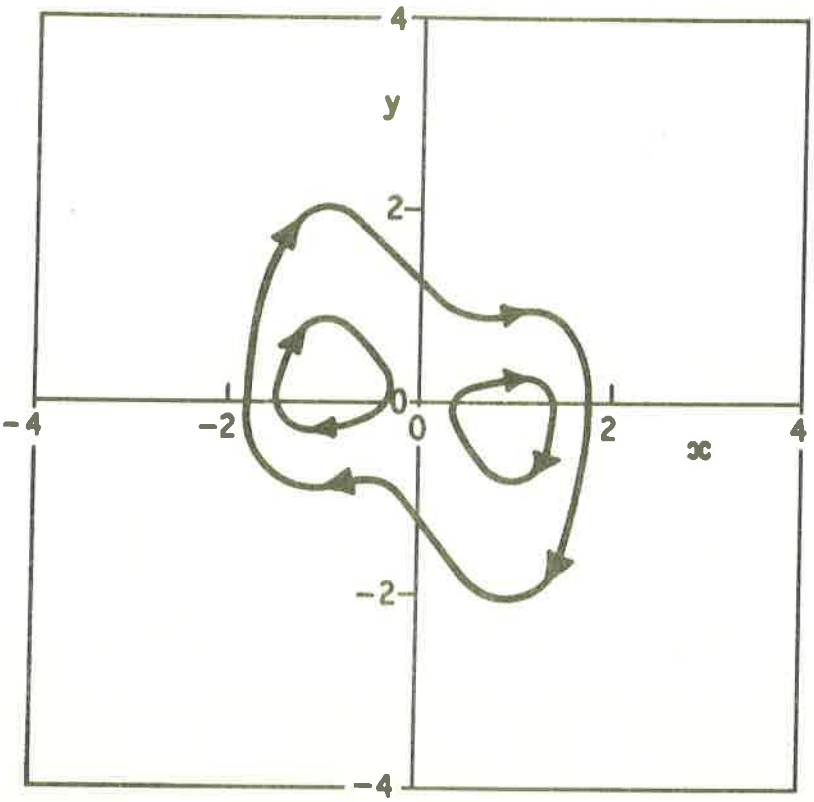

Fig. 22. LIMIT CYCLES $b_0=1, b_2=-1, c_1=-1, c_3=0{\cdot}9$